Second Method:

* $\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x} \therefore \int \frac{1}{1+x} = \int \sum_{n=0}^{\infty}(-1)^n x^n = \ln(1+x)$

Detailed solution!

$$[\ln(1+t)]' = \frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty}(-1)^n t^n$$

$$\Rightarrow \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n\,dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$$

Example 3: Find power series of xe^{-x^2} about $a=0$

Solution! Use $e^{(\,)} = \sum_{n=0}^{\infty}\frac{(\,)^n}{n!}$; so $e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}$

$$\therefore xe^{-x^2} = \sum_{n=0}^{\infty}\frac{x(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!}$$

Extra: Find the 10^{th} derivative of $f(x) = xe^{-x^2}$ at 0

Solution: $\frac{f^{(10)}(0)}{10!}$ = coefficient of x^{10} which is zero (since $2n+1=0$ $n \neq$ an integer)

- $\frac{f^{(11)}(0)}{11!} = \underbrace{\frac{(-1)^5 x^{11}}{5!}}_{\text{coefficient of } x^{11}}$

Question: why is it that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$?

To prove $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$, we must find $S_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$

and $l\, S_n$ must be $f(x)$.

- $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n - S_n(x) =$ error, If l error $=0$, then full series $= f(x)$

- Suppose $\sum_{n=0}^{\infty}a_n(x-a)^n$ and partial sum. They are not equal but the difference between them is the error $= R_n(x)$, if $l\, R_n(x) = 0$, $f(x) =$ series.

Example:

$$S = \sum_{n=1}^{\infty}(-1)^n a_n \quad , a_n > 0, \; a_n \text{ decreasing}, \; l\, a_n = 0$$

$$S_n = \sum_{k=1}^{n}(-1)^k a_k \quad ; |\text{error}, (S - S_n)| \leq a_{n+1}$$

(23SEP2016) Lecture 9:

Error in alternating series

$S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, e.g $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. If we have S_{10}, the $|S - S_{10}| =$ err

- $|S - S_n| \leq a_{n+1}$ (first unused term)

Example 1

$\sum_{n=3}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$, which partial sum can we approximate so that error $< \frac{1}{1000}$?

- Find n

$\hookrightarrow \frac{\ln(n+1)}{n+1} < \frac{1}{1000}$. We know $|S - S_n| < \frac{1}{1000} \Rightarrow |S - S_n| < \frac{\ln(n+1)}{(n+1)} < \frac{1}{1000}$

Example 2:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$, $|\text{error}| < \frac{1}{10^4}$.

- Find n such that $|S - S_n| < \frac{1}{10^4} \Rightarrow |S - S_n| < \frac{1}{(n+1)!}$
- Seek n

$\frac{1}{(n+1)!} < \frac{1}{10^4} \Rightarrow n = 6$, use S_6

- Side note on a challenging problem (unrelated)

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\ln n}{n}}}$; $l \frac{\ln n}{n} = 0 < 1 \Rightarrow \frac{\ln n}{n} < 1$, $n^{\frac{\ln n}{n}} < n$

$\Rightarrow \frac{1}{n^{\frac{\ln n}{n}}} > \frac{1}{n} \therefore \sum_{n=1}^{\infty} \frac{1}{n^{\frac{\ln n}{n}}}$ diverges by DCT

- Given $f(x)$:

$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, the taylor polynomial $P_n(x, a) =$ partial sum

$\therefore \underbrace{|S(x) - P_n(x)|}_{R_n(x)} \leq \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$ ← not a

where $a < c < x$

$\Rightarrow f^{(n+1)}(c) \left| \frac{(x-a)^{n+1}}{(n+1)!} \right| \leq M^{n+1} \left| \frac{(x-a)^{n+1}}{(n+1)!} \right| \leq$ error

Example: In the Taylor expansion of $\cos x$ about $a=0$ [01NOV2016]
if $x=0.1$. Find P_n so that error $< \frac{1}{1000}$

Answer: $$|R_n(x,a)| < \frac{f^{(n+1)}(c)}{(n+1)!}|x-a|^{n+1}$$

Since $f(x)=\cos x$ $\Rightarrow$ $f^{(n+1)}(x) = \pm\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}$ which is always ≤ 1 in absolute value

$\therefore f^{(n+1)}(c) \leq 1 \Rightarrow |R_n(x,0)| \leq \frac{|x|^{n+1}}{(n+1)!}$, so seek n such that

$\frac{(0.1)^{n+1}}{(n+1)!} < \frac{1}{10^3}$ $\therefore n=2$

UNRELATED:

- $\lim \frac{(-1)^n \sin n}{n} = 0$ $\because$ $0 \leq \frac{(-1)^n \sin n}{n} \leq \frac{1}{n}$
- $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\dots\dots$, so if $x>0$ then $e^x > \frac{x^n}{n!}$

choose $x=n \Rightarrow e^n > \frac{n^n}{n!} \Rightarrow n! > \left(\frac{n}{e}\right)^n$

(26SEP2016) Lecture 10

$f(x) = (1+x)^\alpha$ $\alpha=0$, $\alpha \in \mathbb{R}$ (it is a maclaurin series) and is called binomial series)

3 STEPS

1

$f(x)=(1+x)^\alpha$

$f'(x) = \alpha(1+x)^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$

$\vdots$

2

1

α

$\alpha(\alpha-1)$

$\alpha(\alpha-1)(\alpha-2)$

$\vdots$

3

write taylor expansion:

$1+\frac{\alpha}{1!}+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3\dots$

Note: by ratio test, this series converges for $|x|<1$

Example 1: let $\alpha=4$

$(1+x)^4 = 1+4x+\frac{4.3}{2!}x^2+\frac{4.3.2}{3!}x^3+\frac{4.3.2.1}{4!}x^4+0+0\dots\dots$

*In general, if $\alpha=M$, and M is a positive integer, the expansion is finite.

Example 2:

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{(\frac{1}{2})(\frac{1}{2}-1)x^2}{2!} + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

(Alternating series if $x>0$) i.e, if $(1+x)^{\frac{1}{2}} \approx 1+\frac{x}{2}-\frac{x^2}{8}$, then $|\text{error}| < \frac{x^3}{16}$

- How to find Taylor expansion without 3 steps: refer to book chapter 10.10 (The table)
- "Proving" Moivre's formula : $e^{ix} = \cos x + i\sin x$

Use $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

combine with i

- Find all p for which $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)^p$ converges using LCT.

$\therefore$ Need a connection between x and $\sin x$

Answer: From Taylor expansion of $\sin x$:

$$x - \sin x \approx \frac{x^3}{3!} = \frac{x^3}{6} \text{ (for } \underline{\text{small}}\ x)$$

$$\therefore \left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)^p \approx \left(\frac{1}{6n\sqrt{n}}\right)^p, \text{ now use LCT:}$$

$$\rho = \lim\left(\frac{\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}}{\frac{1}{n\sqrt{n}}}\right)^p = \lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)^p = \lim_{x\to 0}\left(\frac{x-(x-\frac{x^3}{3!})}{x^3}\right)^p$$

$$= \lim_{x\to 0}\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} + \cdots\right)^p = \left(\frac{1}{6}\right)^p \text{ which behaves like}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n\sqrt{n}}\right)^p \therefore \sum_{n=1}^{\infty}\left(n^{\frac{3p}{2}}\right)^{-1} \Rightarrow \frac{3p}{2} > 1 \therefore p > \frac{2}{3}$$

(28SEP2016) Lecture 11

Find Taylor series of $f(x) = \frac{1}{1-x}\tan^{-1}(x^2)$ about $a=0$

- For $\tan^{-1}(x) \Rightarrow \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum\limits_{n=0}^{\infty}(-1)^n x^{2n}$

then $\int_0^x \frac{1}{1+t^2}dt = \int_0^x \sum\limits_{n=0}^{\infty}(-1)^n t^{2n}dt = \sum\limits_{n=0}^{\infty}\int_0^x(-1)^n t^{2n} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{2n+1}$

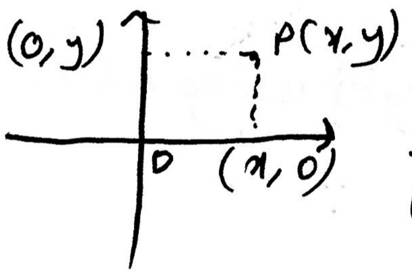
thus $\tan^{-1}(x^2) = \sum \frac{(-1)^n(x^2)^{2n+1}}{} = \sum (-1)^n x^{4n+2}$

ii) $\frac{1}{1-x}\tan^{-1}(x^2) = (1+x+x^2+x^3+\dots)(x^2-\frac{x^6}{3}+\frac{x^{10}}{5}-\dots)$
then distribute term by term.

Theorem: $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$

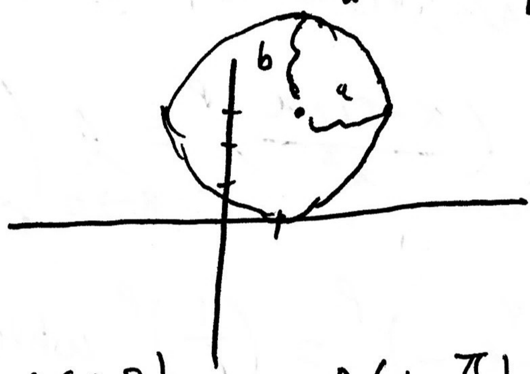
Chapter 11: Polar coordinates

→ Cartesian coordinates:

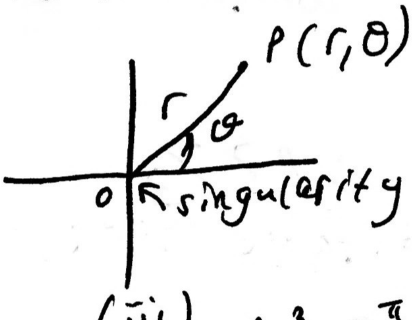


Geometry	Analytic
- Point P	- (x, y)
- Straight line	- set of points governed by a linear equation
- Circle (center, radius) A a	- A(h, k), radius = a i.e $(x-h)^2+(y-k)^2=a^2$
- Parabolas	- $y=ax^2+bx+c$ or $x=ay^2+by+c$
→ Ellipse	→ $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
→ Hyperboloid	→ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

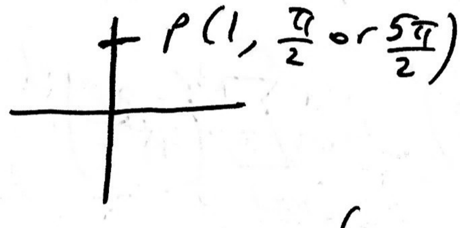
Example:
$$\frac{(x-1)^2}{4}+\frac{(y-3)^2}{9}=1$$



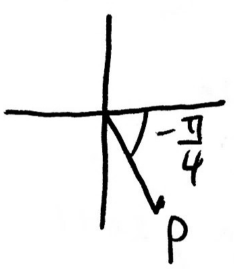
- Polar coordinates of P:



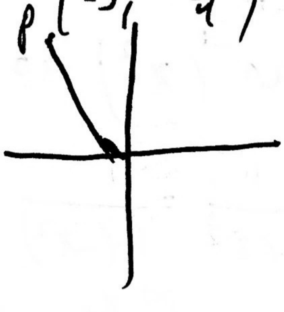
; (i) $P(1, \frac{\pi}{2})$ same as $P(1, \frac{5\pi}{2})$



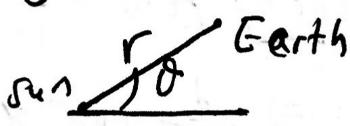
(ii) $P(2, -\frac{\pi}{4})$



(iii) $P(-3, -\frac{\pi}{4})$



§ Some History §



$F \propto \frac{1}{r^2}$, $F = ma$

second derivative of displacement which is r

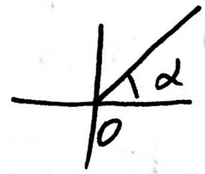
Example 1:

$$r = c$$

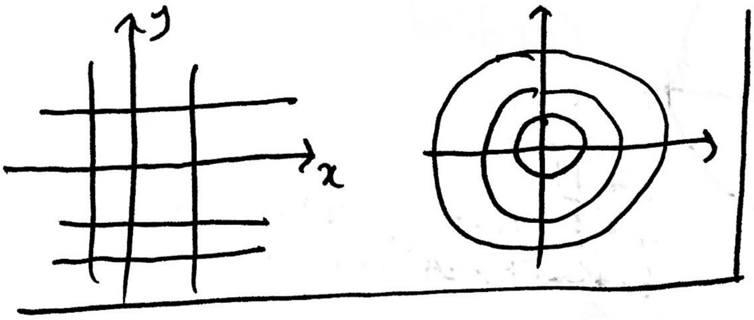
means circle center O radius c

Example 2: $\theta = \alpha$

means ray going through the origin



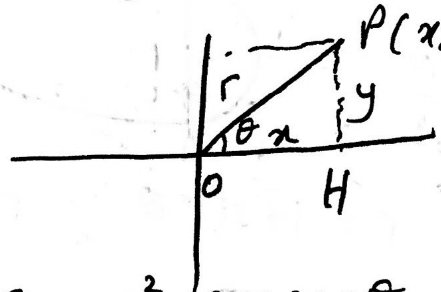
- Cartesian   Polar



Example 3:

$r = 2\cos\theta$

change to Cartesian equation



$$\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x} \; ; \; x \neq 0$$

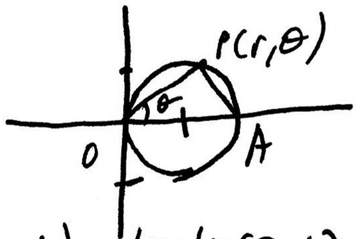
So $r = 2\cos\theta = r^2 = 2r\cos\theta$

$\therefore \; x^2 + y^2 = 2x$

After completing the square:

$(x-1)^2 + y^2 = 1$ (circle center $(1,0)$, $r=1$)

Verification:



$\Rightarrow \cos\theta = \frac{r}{2} \therefore r = 2\cos\theta$!

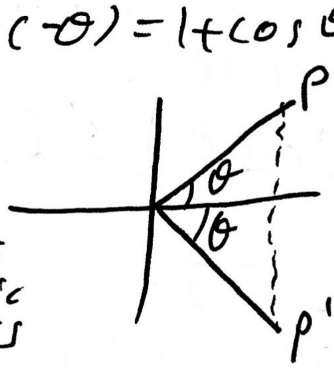
(30 SEP 2016) Lecture 12

Example 1: Sketch $r = 1 + \cos\theta$

First study symmetry!

replace θ by $-\theta$, then $r = 1 + \cos(-\theta) = 1 + \cos\theta$

$\therefore$ points (r,θ) & $(r,-\theta)$ on curve i.e



curve is symmetric to x-axis

- Replace θ by $\pi - \theta$

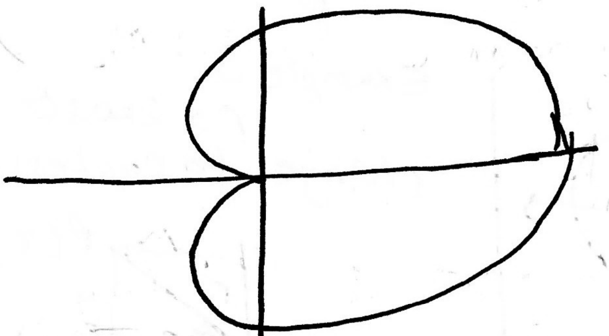
$r = 1 + \cos(\pi - \theta) \Rightarrow r = 1 - \cos\theta$

not symmetric with respect to $\theta = \frac{\pi}{2}$

Table of Values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$1+\cos\theta$	2	~ 1.8	~ 1.7	1.5	1	0.5	~ 0.3	~ 0.2	0

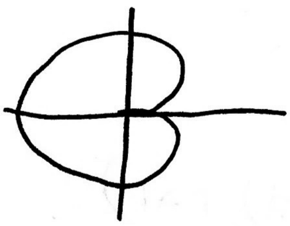
Sketch:



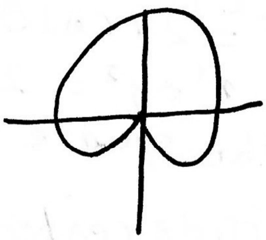
called Cardiod

Similarly:

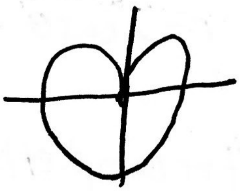
$r = 1-\cos\theta$



$r = 1+\sin\theta$



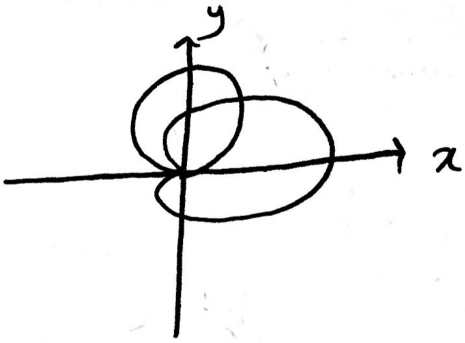
$r = 1-\sin\theta$



Example 2: Sketch:

$r = 2\sin\theta$, $r = 1+\cos\theta$

Find all points of intersection



$r = r$

$2\sin\theta = 1+\cos\theta$

$2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right)$

$\tan\frac{\theta}{2} = \frac{1}{2}$

$\theta = 2\tan^{-1}\left(\frac{1}{2}\right)$

check for $(0,0)$

$\Rightarrow \theta = \pi$, $r=0$, on cardiod

for $\theta = 0$, $r = 0$

$\therefore$ origin is point of intersection

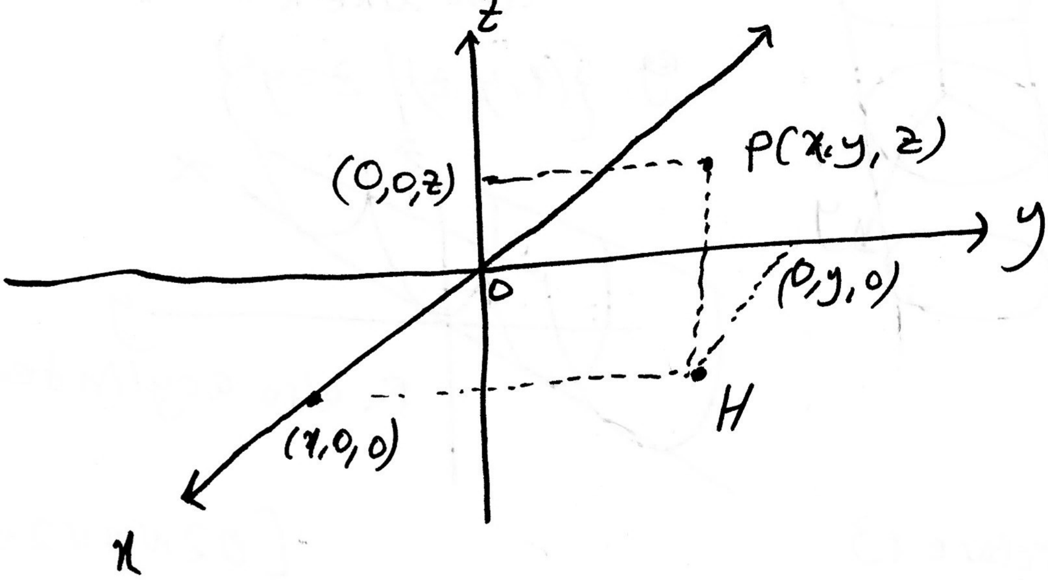
Chapter 12 Quadric Surfaces

In the plane an equation between x and y represent a curve.

e.g 1. $ax+by+c=0$ (2 points needed)

2. $ax^2+ay^2+bx+cy+d=0$ (circle)
need center and radius

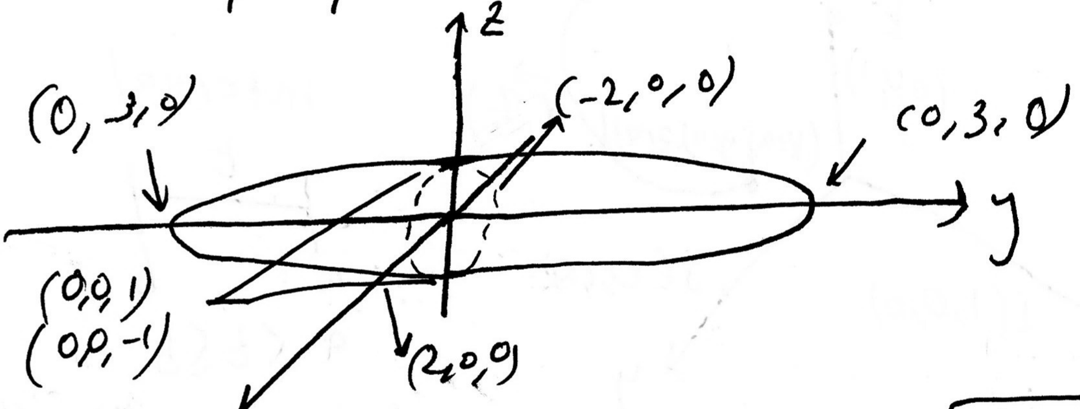
X: In 3d space, a point is given by 3 numbers (x, y, z)



- $(x-h)^2+(y-k)^2+(z-l)^2=R^2$, sphere center (h,k,l) radius $=R$

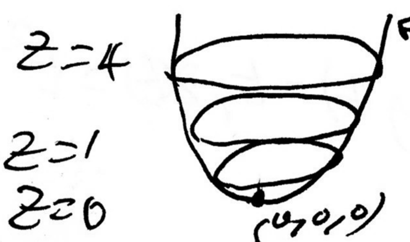
if h, k, l are zero then $x^2+y^2+z^2=R^2$, circle center $(0,0,0)$
radius $=R$

Example 1: $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ (ellipsoid)



Example 2: $z=x^2+y^2$

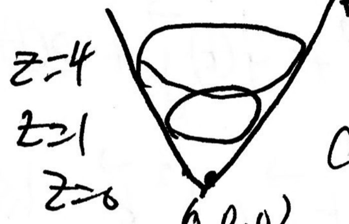
Start "slicing"



parabola (put $x=0$ $\therefore z=y^2$)

called paraboloid

Example 3: $z=\sqrt{x^2+y^2}$



straight line

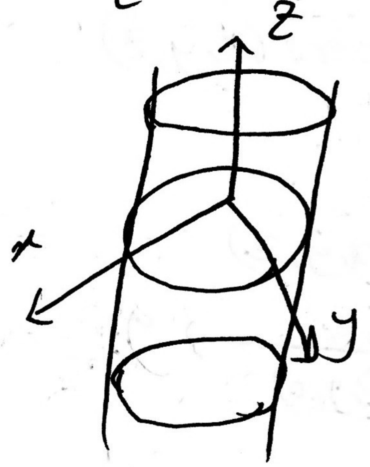
called cone

★Cylinder

In the plane the set of points such that $x^2+y^2=4$ is a circle $C(0,0)$

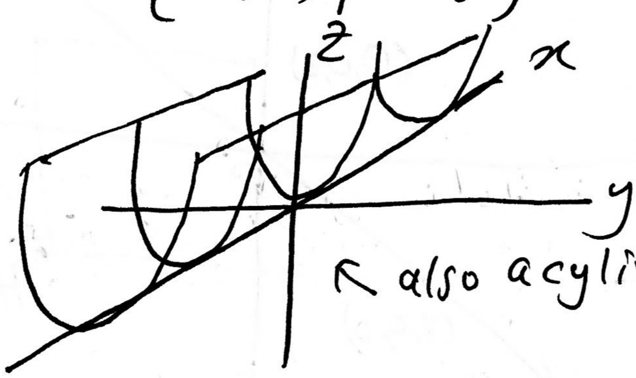
$r=2$, $\{(x,y) \mid x^2+y^2=4\}$

★In space: $\{(x,y,z) : x^2+y^2=4\}$ (base: $x^2+y^2=4$)



Note: a cylinder doesn't have to be close like a soda can.

eg $\{(x,y,z) \mid z=y^2\}$



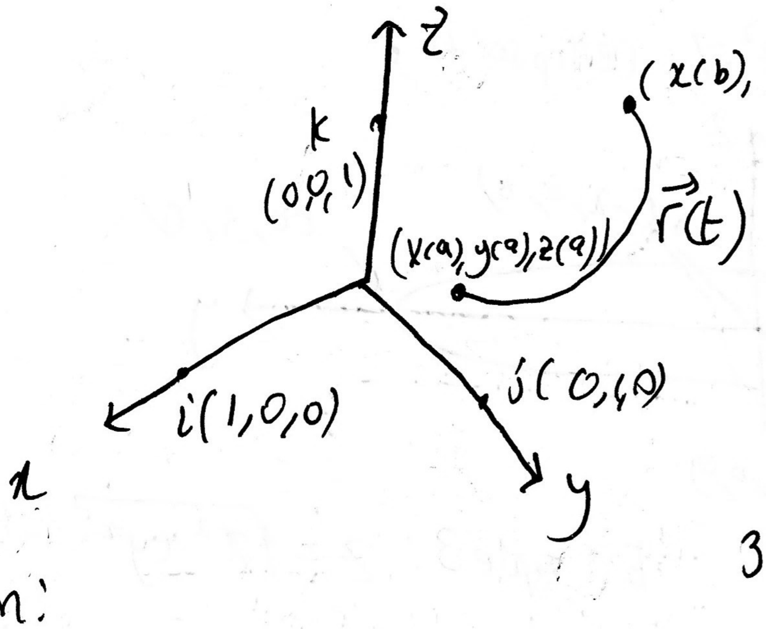
↖ also a cylinder!

(03OCT2016) lecture 13 [02NOV2016]

13.3 & 16.1 Curves in space & line integrals

↓

3d ∴ Domain $\in \mathbb{R}^3$



interval

t

a b

$a \leq t \leq b$

3 functions $x(t)$, $y(t)$, $z(t)$ define a curve in space.

Notation:

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

Example 1:

$\vec{r}(t) = \vec{i} + t\vec{j} + t^2\vec{k}$ $1 \leq t \leq 3$; $x(t)=1$, $y(t)=t$, $z(t)=t^2$

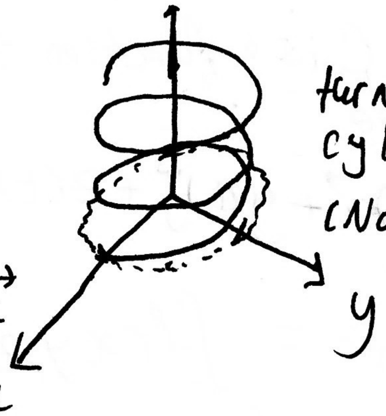
Example 2:

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ $\quad 0 \le t \le 2\pi$

$\left.\begin{array}{l} x(t) = \cos t \\ y(t) = \sin t \end{array}\right\} x^2 + y^2 = 1$ which means the <u>projection</u> of the points on the x-y plane is a circle center $(0,0)$, radius $= 1$

$z(t) = t$

Since $z(t) = t \quad 0 \le t \le 2\pi$ ∴



turns around cylinder ∴ helix
(Note: this is more than 2π)

- $\vec{r}(t)$: position vector
  $\vec{v}(t)$: velocity vector

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k}$$

Examples 3: For helix

$$\vec{v}(t) = (-\sin t)\,\vec{i} + (\cos t)\,\vec{j} + \vec{k}$$

- Arc length of a curve:
- Length: $L = \int_a^b |v(\tau)|\,d\tau$

Example 4: For helix

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,t \Big|_0^{2\pi} = 2\pi\sqrt{2}$$

- Arc length from a to t is denoted $s(t)$

$$s(t) = \int_a^t |\vec{v}(\tau)|\,d\tau$$

Note: speed $= \frac{ds}{dt} = |\vec{v}|$

Example 5: For helix

$$s(t) = \int_0^t \sqrt{2}\,d\tau = \sqrt{2}\,t$$

- $\vec{r}(t) = \vec{i} + t\,\vec{j} + t^2\,\vec{k} \quad 1 \le t \le 3$

Find $\vec{v}(t)$, arc length and speed.

$\vec{v}(t) = \vec{j} + 2t\,\vec{k}$; $|\vec{v}(t)| = \sqrt{1^2 + (2t)^2}$

$$s(t) = \int_1^t \sqrt{1 + 4\tau^2}\,d\tau$$

$v(t)$ is a vector tangent to the curve in space.

- Unit tangent vector

$$\vec{T} = \frac{\vec{V}(t)}{|\vec{v}(t)|} \; ; \quad \text{for helix} \quad \vec{T} = \frac{(-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}}{\sqrt{2}}$$

$$= -\frac{\sin t}{\sqrt{2}}\vec{i} + \frac{\cos t}{\sqrt{2}}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$$

16.1 Suppose C is a function of 3 variables (x, y, z) defined on a curve,

$C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad a \leq t \leq b$

Definition: $$\int_C f\,ds = \int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\underbrace{|v(t)|\,dt}_{ds}$$

Example: For helix

$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k} \quad 0 \leq t \leq \pi, \; f(x,y,z) = xy + z$

Find $\int_C f\,ds$:

- Find $v(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k} \quad 0 \leq t \leq \pi$

$\therefore |v(t)| = \sqrt{2}$ and $f(x,y,z) = f(x(t), y(t), z(t)) = \underset{x}{(\cos t)}\underset{y}{(\sin t)} \underset{+z}{+t}$

$$\Rightarrow \int_C f\,ds = \int_0^\pi (\cos t \sin t + t)\sqrt{2}\,dt = \sqrt{2}\int_0^\pi \left(\frac{\sin 2t}{2} + t\right)dt = \frac{\sqrt{2}}{2}\pi^2$$

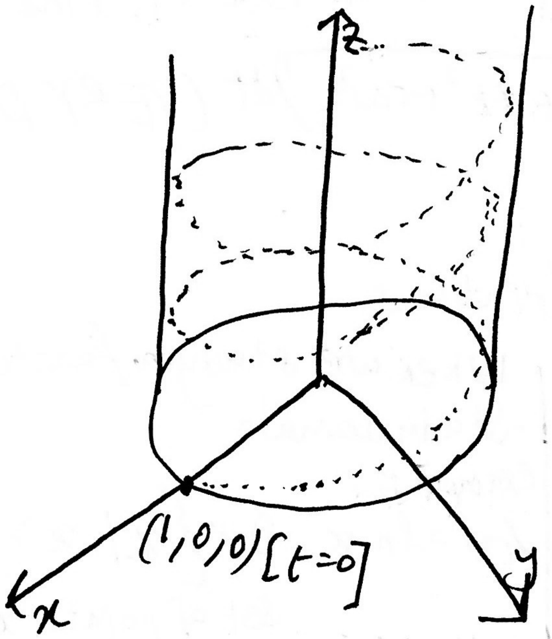
Note: $\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$ (same procedure for dy and dz)

Example: For helix:

$$\int_C f(x,y,z)\,dx = \int_0^\pi (\cos t \sin t + t)\underbrace{(-\sin t)}_{x'(t)}dt = -\frac{\sin^3 t}{3}\Bigg|_0^\pi = -\pi$$

(05OCT2016) Lecture 14

- One way to draw a helix:



- $\vec{r}(t) = (\cos t \sin t)\vec{i} + (\cos^2 t)\vec{j} + (\sin t)\vec{k}$

$x^2 + y^2 = \cos^2 t$

$\therefore x^2 + y^2 + z^2 = 1$

$\therefore$ Sphere with center $(0,0,0)$ & radius $= 1$

- $\vec{F}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$\vec{G}(t) = p(t)\vec{i} + q(t)\vec{j} + l(t)\vec{k}$

Then:

$|\vec{F}(t)| = \sqrt{x^2(t)\vec{i} + y^2(t)\vec{j} + z^2(t)\vec{k}}$

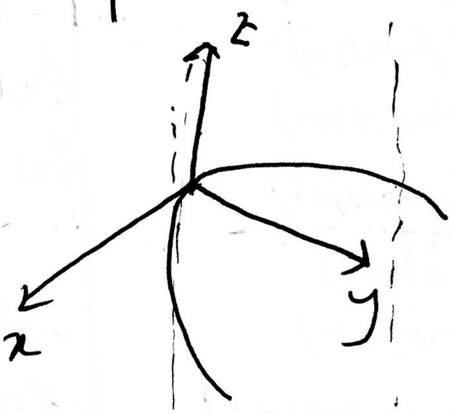
$\vec{F}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$

$\vec{F}(t) \cdot \vec{G}(t) = xp + yq + zl = \text{number}$ ☆

- $\vec{r}(t) = t\vec{i} + t^2\vec{j} + \vec{k}$

$\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} y = x^2$

$z = 1 \leftarrow$ constant



☆ cntd

$$(\vec{F}, \vec{G})'(t) = \frac{d}{dt}(\vec{F}, \vec{G}) = x'p + xp' + y'q + yq' + z'l + zl'$$

$$= \underbrace{(x'p + y'q + z'l)}_{F'(t) \cdot G(t)} + \underbrace{(xp' + yq' + zl')}_{G'(t) \cdot F(t)}$$

- $\vec{F} \times \vec{G} = \begin{vmatrix} i & j & k \\ x & y & z \\ p & q & l \end{vmatrix} = (yl - qz)\vec{i} - (xl - pz)\vec{j} + (xq - yp)\vec{j}$

★ Research: Prove that $\frac{d}{dt}(\vec{F} \times \vec{G}) = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$

- Application of line integrals:

- Mass of $C = \int_C \delta(x, y, z)\,ds$ where $\delta(x, y, z)$: density

- Moments: $\int_C x\,ds$ (first moment), $\int_C y\,ds$ (second moment)

$\int_C z\,ds$ (third moment). These can be use to find center of mass

- work
- flow
- circulation etc...

Example: A wire of ρ: $\delta(x,y,z) = xy + 1$ is bent in the form of a curve of equation $C: \vec{r}(t) = t\,\vec{i} + t^2\vec{j} - \sin t\,\vec{k}$ $0 \leq t \leq 1$. Find Mass

Answer: $\int_C \delta(x,y,z)\,ds = \int_0^1 t(t^2)+1\left(\sqrt{1^2+4t^2+\cos^2 t}\right)dt$ (VERY DIFFICULT!)

(07OCT2016) Lecture 14

Function of Several Variables

Review: Function of 1 variable:

- Variable: $x, t, \ldots$
- function: $f, g, h, p \ldots$
- value of f at x: $f(x)$

eg $f(x) = 5$ (constant function)

$f(x) = x^n$ (power function)

$f(x) = 2x^2 - 3x$ (polynomial)

$f(x) = e^x$ (exponential)

$f(x) = \sin x$ (trigonometric)

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ (power series)

When we study a functions:

- Study domain

examples:

$f(x) = \ln x$. $D = \{x \mid x > 0\}$

Set of points x such that x is greater than zero

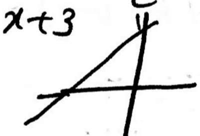
$f(x) = \sqrt{x-5}$. $D = \{x \mid x \geq 5\}$

$f(x) = \frac{1}{\sqrt{x^2-4}}$. $D = \{x \mid x^2 - 4 > 0\}$

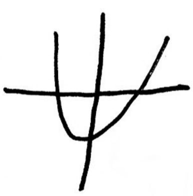
$D = \{x \mid x < -2, x > 2\}$

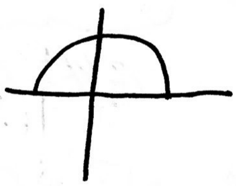
- Study Variations i.e derivatives & signs
- Graph of f i.e $\{(x,y) \mid y = f(x)\}$

i.e. $f(x) = x + 3$

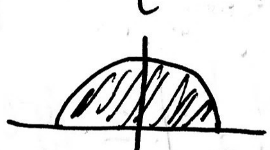


- $f(x) = x^2 - 3x - 4$ • $f(x) = \sqrt{4-x^2}$





Extra: Graph $\{(x,y) \mid y \leq \sqrt{4-x^2}, y \geq 0\}$



- Limits and continuity
- Differentiability
- Integration

Functions of 2 variables

- 2 variables: $(x,y), (s,t), (u,v) \ldots$
- function: $f, g, h \ldots$
- Value of a function at $(x,y) = f(x,y)$

- Examples:

$f(x,y) = 5$

$f(x,y) = x^n y^m$, $x^n + y^m \ldots$

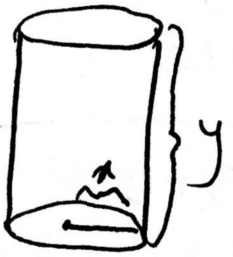
$f(x,y) = x^2y^3 - 6xy^5 + 1$

$f(x,y) = e^{x+y}$, e^{xy}, $e^{x^2y^{100}}$ ---

$f(x,y) = \sin(xy)$, $\sin(x+y)$ ---

- Uses of 2 variables: [04NOV2016]

Example:



$V = \pi x^2 y$ (volume of a cylinder is dependant on 2 variables)

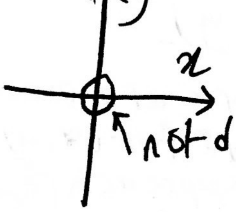
- Domain of $f(x,y)$

$D = \{(x,y) \mid f(x,y) \text{ is defined}\}$

e.g. $f(x,y) = \ln(x^2+y^2)$. Find domain of $f(x,y)$

- $D = \{(x,y) \mid x^2+y^2 > 0\} \Rightarrow \{(x,y) \mid (x,y) \neq (0,0)\}$

or $\mathbb{R}^2 - \{(0,0)\}$

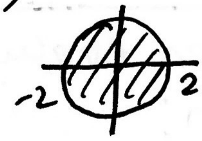


not defined at (0,0)

Another example:

$f(x,y) = \sqrt{4-x^2-y^2}$. Find domain of $f(x,y)$

$D = \{(x,y) \mid 4-x^2-y^2 \geq 0\}$ which means $x^2+y^2 \leq 4$ (disc center (0,0), radius = 2)



- Graphs:

Given $f(x,y)$, graph is $\{(x,y,z) \mid z = f(x,y)\}$

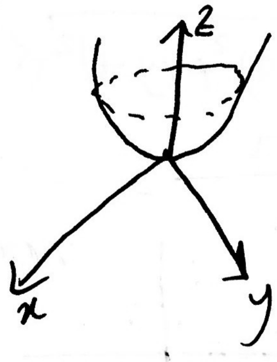
i.e. a) $f(x,y) = x^2+y^2$

b) $f(x,y) = \sqrt{x^2+y^2}$

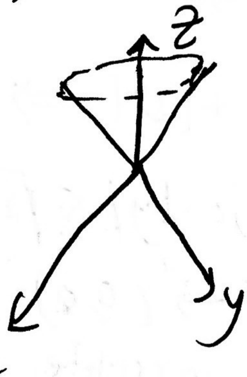
c) $f(x,y) = \sqrt{4-x^2-y^2}$

Solutions!

a) $z = x^2+y^2$

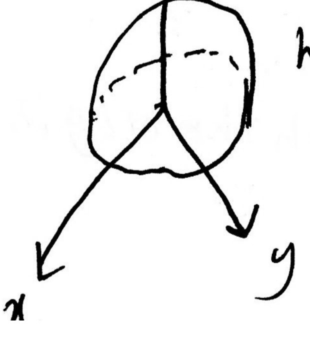


b) $z = \sqrt{x^2+y^2}$



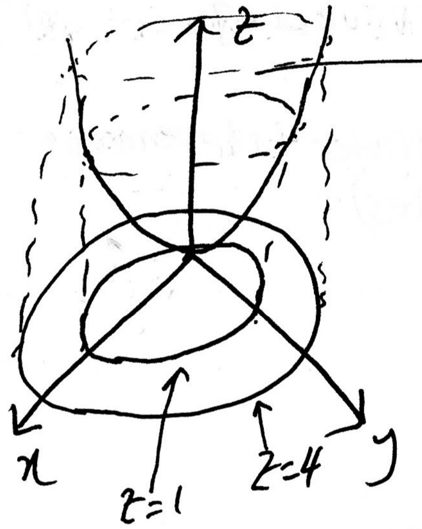
c) $z = \sqrt{4-x^2-y^2}$ $z \geq 0$

$\wedge$ $z^2 + x^2 + y^2 = 4$

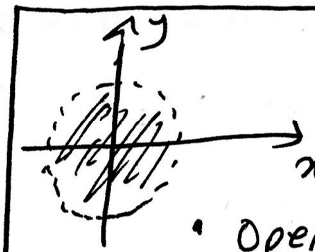


hemisphere

Note: In general, if given $f(x,y)$ then $z = f(x,y)$. Look at curves when z is a constant, such curves are called level curves.



Finally neighbourhoods (nbhd) and <u>open</u> sets.



$$\{(x,y) \mid \sqrt{(x-a)^2+(y-b)^2} < \delta\}$$

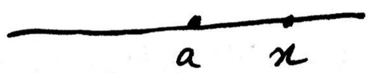
where (a,b) is a point in the disc. (The center)

- Open disc, center (a,b), radius $=\delta$
  $\Rightarrow$ nbhd (a,b)

(10 OCT 2016) Lecture 15

Limits and Continuity

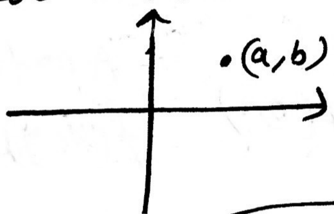
In one variable:



$x \to a$ i.e $|x-a| \to 0$

$$\lim_{x \to a} f(x) = L$$

In two variables:



* There are many ways for (x,y) to approach (a,b)

- $(x,y) \to (a,b)$ i.e $\sqrt{(x-a)^2+(y-b)^2} \to 0$

Advice: Use these inequalities when evaluating limits with 2 variables.

- If u and v are real numbers, then:

$|u| = \sqrt{u^2} \leq \sqrt{u^2+v^2}$; $|v| = \sqrt{v^2} \leq \sqrt{u^2+v^2}$

$\Downarrow$ $\Downarrow$

$|x-a| \leq \sqrt{(x-a)^2+(y-b)^2}$ $|y-b| \leq \sqrt{(x-a)^2+(y-b)^2}$

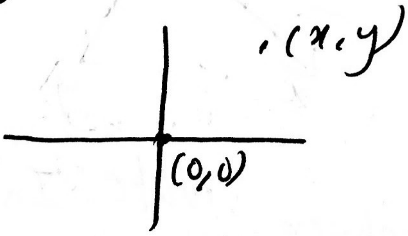
Also:

$\sqrt{u^2+v^2} \leq |u|+|v| \Rightarrow \sqrt{(x-a)^2+(y-b)^2} \leq |x-a|+|y-b|$

- $|x| \leq \sqrt{x^2+y^2}$; $|y| \leq \sqrt{x^2+y^2}$

Example: $(x,y) \to (0,0)$

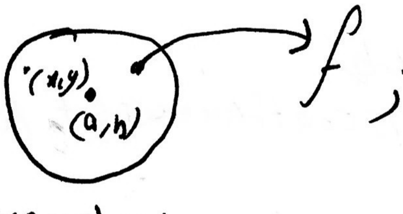
What are the possible paths of $(x,y) \to (0,0)$?



- Infinitely many, i.e: $y = x$, $y = 2x$, $x = y^2$, $y = x^2$ etc... } all pass through the origin

- Let f be a function of 2 variables defined in a nbhd of the point (a,b), except possibly at (a,b):

i.e.



f has limit L as $(x,y) \to (0,0)$ if to every $\varepsilon > 0$ there is a $\delta > 0$, such that:

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \text{ then } |f(x,y) - L| < \varepsilon$$

Examples:

- $\lim_{(x,y)\to(1,2)} x+y = 3$; $\lim_{(x,y)\to(1,2)} \sin(xy) = \sin 2$

- Let $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ $(x,y) \neq 0$

Find domain of f and $\lim_{(x,y)\to(0,0)} f(x,y)$ if the limit exists.

- Domain $= \{(x,y) \mid (x,y) \neq (0,0)\}$
- Before deciding on $\lim$, consider different methods of approach:

i.e:

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^3}{2x^2} = \lim_{x\to 0} \frac{x}{2} = \frac{0}{2} = 0 \quad (\text{when } y = x)$$

Yet this result is NOT enough to prove the limit exist, thus other paths must be tried.

i.e.

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{x^2+x^4} = \lim_{x\to 0} \frac{x^2}{1+x^2} = \frac{0}{1+0} = 0$$

Since we are suspecting that the limit is indeed zero, look at:

$$|f(x,y) - 0| = \left|\frac{x^2 y}{x^2+y^2}\right| = \frac{x^2|y|}{x^2+y^2} \leq \frac{x^2 \cdot \sqrt{x^2+y^2}}{x^2+y^2} \leq \frac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2}$$

$= \sqrt{x^2+y^2}$, but $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = \sqrt{0+0} = 0$ $\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$ by CST*

* Collorary to the sandwich theorem.

- Let $f(x,y) = \dfrac{xy}{x^2+3y^2}$ $(x,y) \neq (0,0)$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists

- If $y = x$, $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \dfrac{x^2}{4x^2} = \dfrac{1}{4}$
- If $y = x^2$, $\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \dfrac{x^3}{x^2+3x^4} = \lim_{x\to 0} \dfrac{x}{1+3x^2} = 0$

$\Big\}$ limit doesn't exist

Continuity

Let f be defined in a nbhd of (a,b) [including (a,b)] such that $f(a,b)$ is defined. Also, $f(x,y)$ if $\lim\limits_{(x,y)\to(a,b) \equiv (x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b)

Example:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 5 & (x,y) = (0,0) \end{cases}$$

- $f(0,0) = 5$ (given)
- Find $\lim\limits_{(x,y)\to(a,b)} f(x,y) \Rightarrow$ found to be zero in the previous example.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) \Rightarrow f$ is not continuous at $(0,0)$

Rules: (also found in book)

$$\lim(f+g) = \lim f + \lim g \qquad \lim f^n = (\lim f)^n$$

$$\lim(fg) = (\lim f)(\lim g) \qquad \lim (f)^{\frac{1}{n}} = (\lim f)^{\frac{1}{n}} \quad (f>0)$$

$$\lim\left(\frac{f}{g}\right) = \frac{\lim f}{\lim g} \quad (\lim g \neq 0)$$

- If f and g are continuous at (a,b) then: $f+g$ is continuous at (a,b), likewise for fg, $\frac{f}{g}$, f^n, $f^{\frac{1}{n}}$.
- All power functions are continuous everywhere eg $x^n y^m$
- All polynomials are continuous.
- $\sin(x^n y^m)$, $\cos(xy)$, e^{x-y}, $\ln(x^2+y)$ .... are all continuous.

Example:

$$f(x,y) = \begin{cases} x^2+y^2 \sin \dfrac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that $f(x,y)$ is continuous at $(0,0)$: $f(0,0) = 0$ (Given)

$$* \left| x^2+y^2 \sin\frac{1}{x^2+y^2} - 0 \right| = \left| \frac{\sin \frac{1}{x^2+y^2}}{\frac{1}{x^2+y^2}} \right| = \left| \frac{\sin t}{t} \right| \leq \boxed{\frac{1}{t}} \text{ goes to zero}$$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ is continuous at $(0,0)$

(1140 T2016) Lecture 16

$f(x,y) = \dfrac{x\sin y}{x^2+\sin^2 y}$ $(x,y) \neq (0,0)$. Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

- Along $y=0$, $\lim\limits_{x\to 0} f(x,0) = \dfrac{0}{x^2+0} = 0$
- Along $y=x$, $\lim\limits_{x\to 0} f(x,x) = \dfrac{x\sin x}{x^2+\sin^2 x}$

$= \lim \dfrac{x^2\left(\frac{\sin x}{x}\right)}{x^2\left(1+\frac{\sin^2 x}{x^2}\right)} = \dfrac{1}{1+1^2} = \dfrac{1}{2}$

$\frac{1}{2} \neq 0$

$\therefore$ limit doesn't exist.

Remark: During the lecture a student suggested we put $x = \cos y$, since $\cos^2 y + \sin^2 y = 1$ however we can't do that because for $y=0$, $\cos 0 = 1$ which does NOT pass through the origin.

Partial derivatives: (First a little review on one variable)

$\dfrac{d}{dx}(yx^n) = nyx^{n-1}$ (x, y are independant)

$\dfrac{d}{dx}(\sin(xy)) = y\cos(xy)$

Remember! $f'(x_0) = \dfrac{f(x_0+h) - f(x_0)}{h}$

Now for $f(x,y)$ and a point (x_0,y_0), how do we differentiate?

3 options:

Fix y, study x or fix x, study y or study both x & y.

- Notation and naming:

$f_x(x_0,y_0) = \dfrac{\partial f}{\partial x}(x_0,y_0)$ is read as partial derivative of f with respect to x evaluated at the point (x_0,y_0)

∂ read as "del"

and defined as: $\lim\limits_{h\to 0} \dfrac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$

Example: $f(x,y) = x^n y^m$, Find $\dfrac{\partial f}{\partial x}(1,2)$

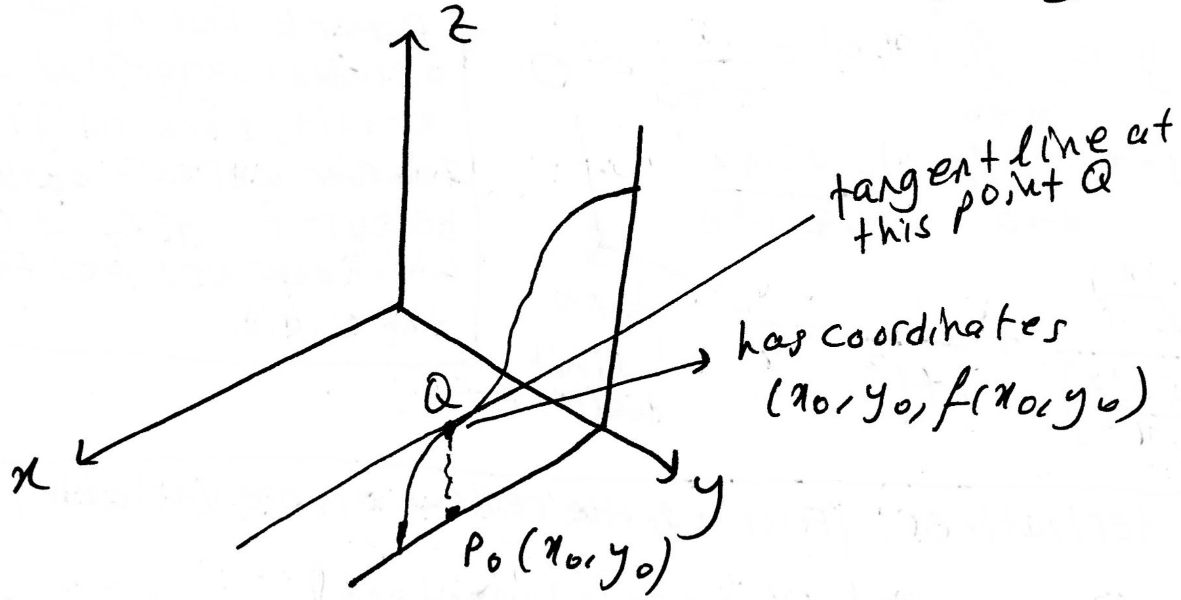
Solution: $\dfrac{\partial f}{\partial x} = nx^{n-1}y^m$ so $\dfrac{\partial f}{\partial x}(1,2) = n2^m$

Note: all what was mentioned is similar for $\dfrac{\partial f}{\partial y}$, only this time x is being treated as a constant and y as the only variable.

- Geometric Interpretation:

Given $f(x,y)$ and a point $P_0(x_0,y_0)$, interpret $\frac{\partial f}{\partial x}$ at P_0.

Solution: sketch the graph of f i.e surface $z=f(x,y)$

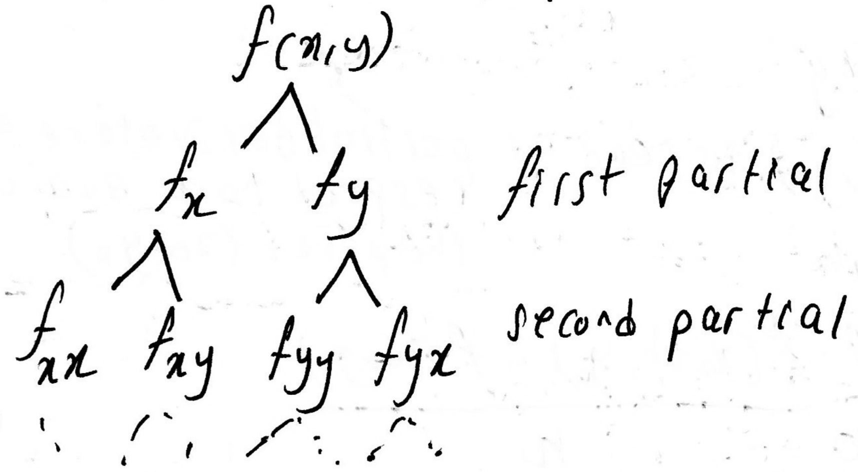


Interpretation:

$\frac{\partial f}{\partial x}(P_0)$ is the slope of the tangent to the curve obtained by intersection of the graph $z=f(x,y)$ with the plane $y=y_0$

* The case is similar for $\frac{\partial f}{\partial y}$ (i.e $x=x_0$)

- What about higher partials?



Euler's Theorem: If the partials of the first 2 order are continuous themselves, then the mixed partials are equal.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$