[27MAY2019] 3:02 P.M.

It seemed good to me to share my notes to those who will take MATH201. You will notice throughout 2 different dates; the one in square brackets (i.e. [ ]) represents the date I wrote (in fact, copied) my notes; the ones in parentheses that is, ( ), represents the day of the lecture three years ago.

She has forgotten me, so ye people, remember me...

Best.

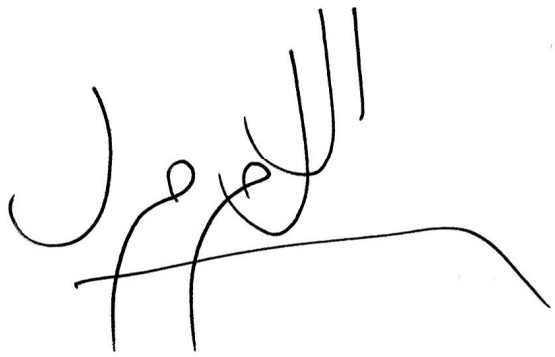
"As much as I want you, I cannot have you."

M.L.

"لاموني في حُبّي لَيلي بَلْ مِنْ سِوَهَا قَلْبِي كَمِدُ"

م.ل.

"나는 아직도 그대가 보고픈데
나는 지금도 그대가 그리운데..." ㅁ.ㄹ.



[28 OCT 2016]

Math 201 Class Notes:

This is my math 201 notes which I have organised and included my own commentary & hints. The dates I will write down, are the dates in which I had the lecture. I hope this finds you well.

Lecture 1 — Chapter 10 — (31 Aug 2016)

- Sequences

We learn this to understand infinite series and eventually power series for us to answer questions such as:

- Is $0.9999\ldots = 1$?
- $e^{ix} \stackrel{?}{=} \cos x + i \sin x$

or some other problems like:

Define $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$

Prove $e^x > \frac{x^n}{n!}$

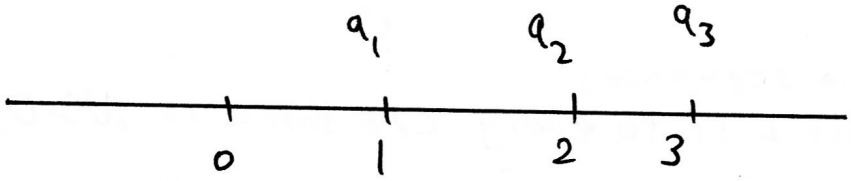
- Definition of a sequence:

A sequence is a list of numbers, denoted as:

$a_1, a_2, a_3, \ldots$

1st term, 2nd term, 3rd term

- A sequence is a function of positive integers set:



Example 1:

$6, 8, 10, 12 \ldots$

This sequence can be represented as $a_n = 2n + 6$ from $n = 0$

or $a_n = 2n + 4$ from $n = 1$

Example 2:

$5, 5, 5, 5 \ldots$ (called a constant sequence)

i.e $a_n = 5$

Example 3:

$1, -1, 1, -1, 1, \ldots$

$a_n = -\cos(n\pi)$ from $n = 1$ or $a_n = (-1)^{n+1}$ from $n = 1$

Example 4:

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$

$b_n = \frac{1}{n}$ from $n = 1$

Example 5:

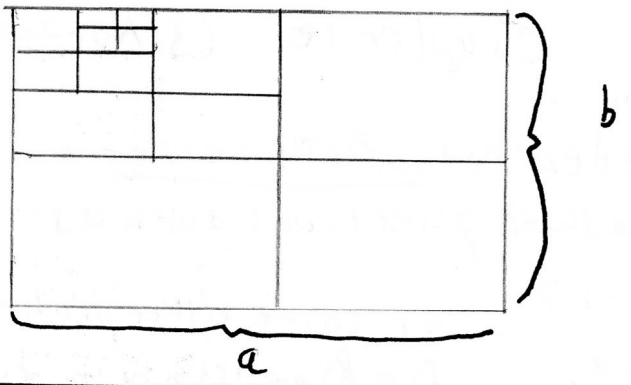
$$a_{n+2} = a_{n+1} + a_n \quad \text{(Fibonacci Sequence)}$$

where $a_0 = 0$ and $a_1 = 1$, then:

0, 1, 1, 2, 3, 5, 8... (recurrence relation)

Example 6:

Note!
Figure NOT accurately drawn.



$A_1 = ab$

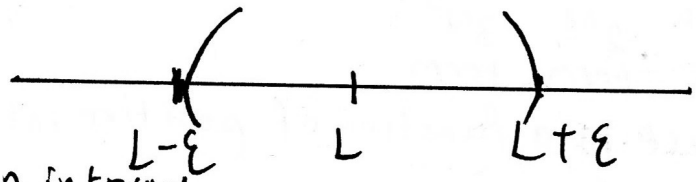
$A_2 = (ab)\frac{1}{2^2}$

$A_3 = (ab)\frac{1}{2^3}$

$\vdots$

$A_n = (ab)\frac{1}{2^n}$ from A_2

Limit of a sequence:
- Given a sequence a_n, number L is a limit of this sequence if for a given positive number ε (epsilon) the terms of the sequence "live" in the interval $(L-\varepsilon, L+\varepsilon)$ after some number.



Thus $-\varepsilon < a_n - L < \varepsilon$ if a_n is in interval.

$\therefore \quad 0 \leq |a_n - L| < \varepsilon$

Definition of limit of a sequence:
- sequence a_n has limit L if to every $\varepsilon > 0$ there is $N > 0$, such that if $n > N$ then $|a_n - L| < \varepsilon$

n: all number greater than N; N: some arbitrary number (not necessarily from the beginning)

- a_n is convergent to α or $\lim_{n \to +\infty} a_n = L$

Example 1:

$$\lim_{n \to +\infty} \frac{1}{n} = 0$$

Proof:

Suppose $\left|\frac{1}{n} - 0\right| < \varepsilon$ then $\frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$,

Given $\varepsilon > 0$, take $N > \frac{1}{\varepsilon}$ and fix it. So if $n > N$ then $n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$

$\therefore \quad \left|\frac{1}{n} - 0\right| < \varepsilon$ Q.E.D

Application:

If $\lim a_n = a$
$\lim b_n = b$ (ofcourse as $n \to +\infty$)

then,

① $\lim (a_n + b_n) = a+b$

② $\lim (a_n b_n) = ab$

③ $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ provided $b_n \neq 0$; $b \neq 0$

Proof of ①:

$$|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

(Property: $|a+b| \leq |a| + |b|$) e.t.c...

Example 2:

$$\lim_{n \to +\infty} \frac{1}{n^2} = \lim \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = 0.0 = 0$$

Lecture 2: (02SEP2016)

What we will learn → Practical: computing limits
→ Theoretical: How to know whether the sequence is converging without calculating limit.

Theorem: If a_n is a monotone sequence (that is, strictly increasing or strictly decreasing) which is bounded above/below, then a_n is convergent

Terms to know:

- strictly increasing $\Rightarrow a_n < a_{n+1}$
- nondecreasing $\Rightarrow a_n \leq a_{n+1}$

- If a_n is bounded from above there is a constant M (independent of any n) such that:

$$a_n \leq M \text{ for all } n$$

Example 1:

$$a_n = 1 - \frac{1}{n}$$

Observe $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$$

$\therefore a_{n+1} > a_n$

$\Rightarrow a_n < a_{n+1}$ (bounded)

$\therefore a_n \leq 1$ for all n $\Rightarrow a_n$ is convergent.

- Sandwich theorem:

Given 3 sequences $a_n \leq b_n \leq c_n$

If $\lim_{n \to +\infty} a_n = L = \lim_{n \to +\infty} c_n$ then $\lim_{n \to +\infty} b_n = L$

Proof:

Suppose $|a_n - L| < \varepsilon$ for $n > N_1$

$|c_n - L| < \varepsilon$ for $n > N_2$

$|b_n - L| < \varepsilon$ for $n > N_1$ or N_2 depending on whether $N_1 > N_2$ or $N_2 > N_1$ $\Rightarrow$ choose larger N

Note: N_1 and N_2 can be the same.

Example 1:

$$-\frac{1}{n} < \frac{(-1)^n}{n} < \frac{1}{n} \quad (\because -1 \leq (-1)^n \leq 1$$

$\lim_{n\to+\infty} -\frac{1}{n} = 0$ and $\lim_{n\to+\infty} \frac{1}{n} = 0 \therefore \lim_{n\to+\infty} \frac{(-1)^n}{n} = 0$ by Sandwich Theorem (S. T)

Example 2:

$$\lim_{n\to+\infty} \frac{\cos(n^3)}{n} = 0 \quad (\because -1 \leq \cos(\text{anything}) \leq 1 \;;))$$

Note: I will evaluate $\lim_{n\to+\infty} \frac{x^n}{n!}$ later in the Appendix I [1]

- If f is continuos at L, and a_n is a sequence converging to L, then $\lim_{n\to+\infty} f(a_n) = f(L)$

Example 1:

Given $\lim_{x\to 0} \frac{\sin x}{x} = 1$ [2] & $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$ (use S. T)

Then $\lim_{n\to+\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$; for $\lim_{n\to+\infty} \frac{1}{n} = 0$ (similar for $\frac{\ln n}{n}$ instead of $\frac{1}{n}$)

Example 2:

$$\lim_{n\to+\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x\to 0} \frac{e^x - 1}{x} = 1 \quad [3]$$

Practice Questions

Find

$$\lim_{n\to+\infty} (n^{1+\frac{1}{n}}) \sin(\tfrac{1}{n})$$

Note: I shall denote $\lim_{n\to+\infty}$ as ℓ

Solution:

$$\ell (n^{1+\frac{1}{n}}) \sin(\tfrac{1}{n}) = \ell\, n \cdot n^{\frac{1}{n}} \cdot \sin\tfrac{1}{n} = \ell n^{\frac{1}{n}} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \ell\, n^{\frac{1}{n}} \cdot \ell \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \cdot 1 = 1$$

- Given $a_n = \sqrt{n+1} - \sqrt{n}$

Find $\ell\, a_n$

Solution: $\ell\, a_n = \ell(\sqrt{n+1} - \sqrt{n}\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \ell \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$

- Find $\ell\,(n+5)^{\frac{1}{n}}$

Solution:

Let $y=(n+5)^{\frac{1}{n}}$ then, $\ln y=\frac{\ln(n+5)}{n}$

$\Rightarrow \ell \ln y=\ell\,\frac{\ln(n+5)}{n}=0$

$\therefore \ell\, y=e^{0}=1$

- Find $\ell\left(\frac{n-5}{n+6}\right)^{n}$.

Solution: $\ell\left(\frac{n-5}{n+6}\right)^{n}=\ell\left(\frac{n(1-\frac{5}{n})}{n(1+\frac{6}{n})}\right)^{n}=\frac{e^{-5}}{e^{6}}=e^{-11}$

- Find $\ell\,\frac{1}{2^n n!}$

Solution: $\ell\,\frac{1}{2^n n!}=\frac{(\frac{1}{2})^n}{n!}=0$

Note:

$\ell\,\frac{\ln n}{n^{(\text{any positive power})}}=0$

$\because \ln n<n^{\alpha}$ for $n>n_0$

* Infinite series:

Given $x>0$

$$1-x^3=(1-x)(1+x+x^2)$$
$$1-x^4=(1-x)(1+x+x^2+x^3)$$
$$\vdots$$
$$1-x^n=(1-x)(1+x+x^2+\dots+x^{n-1})$$

or

$$1-x^{n+1}=(1-x)(1+x+x^2+\dots+x^n)$$

If $x\neq 1$ $(1-x\neq 0)$ then:

$1+x+x^2+\dots+x^n=\frac{1-x^{n+1}}{1-x}$; if $|x|<1$ or $-1<x<1$ then $1+x+x^2\dots=\frac{1}{1-x}$

e.g.

If $x=-\frac{1}{3}$, then:

$$1+(-\tfrac{1}{3})+(-\tfrac{1}{3})^2+\dots+(-\tfrac{1}{3})^n=\frac{1-(-\frac{1}{3})^{n+1}}{1-(-\frac{1}{3})}$$

Application of formula:

Find $\ell\left[1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^n}\right]$

Solution:

$$1+\tfrac{1}{2}+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3+\dots+(\tfrac{1}{2})^n=\frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}}=2(1-\tfrac{1}{2^{n+1}})$$

$\therefore \ell\, 2(1-\frac{1}{2^{n+1}})=2$

Note on notation:

$a_1 + a_2 + a_3 \cdots$ ← 3 dots mean an <u>infinite</u> sum

To understand this we must understand what is meant by a partial sum which is denoted by S_n.

$$\left.\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}\right] \text{ called sequence of partial sums.}$$

Hence definition of the full sum

$$\lim_{n} S_n = S \qquad (a_1 + a_2 + a_3 + \cdots)$$

Lecture 3 (05SEP2016)

The partial sum is also expressed as: $\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Solution:

The nth partial sum, S_n, is $a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{n+1}$ [4]

So $\lim S_n = 1 \therefore$ sum $= 1$

General Rule: The infinite sum $\sum_{n=1}^{\infty} a_n$ is the $\lim S_n$.

- Geometric Series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1}$$

where the nth partial sum, $S_n = a + ar + ar^2 + \cdots ar^n$

$$= a(1 + r + r^2 + \cdots + r^n) = \frac{a(1 - r^{n+1})}{1 - r}$$

If $|r| < 1 \Rightarrow \frac{a}{1-r}$

- Some terms to know:
  - a_1, a_2, a_3 are called terms
  - S_n is called (n^{th}) partial sum

* Sometimes it's not possible to compute $S_n = \sum_{n=1}^{N} a_n$, but there are several test to determine whether the series converges or diverges

I. The nth term divergence test:

Given $\sum_{n=1}^{\infty} a_n$, if $\lim a_n \neq 0$ or doesn't exist, then series diverges.

Proof:

First prove that if $\sum_{n=1}^{\infty} a_n$ converges <u>then</u> $\lim a_n = 0$

- Suppose $\sum_{n=1}^{\infty} a_n$ converges, then S_n has a limit S, and also S_{n-1} has a limit S.

$\therefore \lim (S_n - S_{n-1}) = S - S = 0$, but $S_n - S_{n-1} = a_n$ [5]

$\therefore \lim a_n = 0$

Hence if $\lim a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges ($p \supset q$, $\sim q$, $\therefore \sim p$ [6])

Theorem:

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are convergent then:

(i) $\sum_{n=1}^{\infty} a_n + b_n$ is also convergent (same as $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(ii) $\sum_{n=1}^{\infty} c a_n$ is also convergent, where c is a constant <u>not</u> involving n.

Example:

$$\sum_{n=1}^{\infty}\left[\frac{1}{n(n+1)} + \frac{(-1)^n}{2^{n+1}}\right] = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}}}_{\frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n \text{ (ii)}} \quad \text{(i)}$$

Proof of (i)

Suppose we have:

$$S_n = \sum_{k=1}^{n} a_k \quad , \quad t_n = \sum_{k=1}^{n} b_k$$

(S_n has limit S) (t_n has limit t)

$\therefore \lim (S_n + t_n) = S + t$

$\Rightarrow S_n + t_n = \sum_{k=1}^{n} a_k + b_k = n^{th}$ partial sum of $\sum_{n=1}^{\infty} a_n + b_n$

$\therefore$ series is convergent

Proof of (ii)

$$\sum_{n=1}^{\infty} c a_n = c a_1 + c a_2 + c a_3 + \ldots = c(a_1 + a_2 + a_3 \ldots) = c\sum_{n=1}^{\infty} a_n$$

Lecture 4 (07SEP2016)

More Problems

$-\ell\,(3^n+8^n)^{\frac{1}{n}}$

Apply S.T

$$8^n < 3^n + 8^n < 8^n + 8^n$$

$$(8^n)^{\frac{1}{n}} < (3^n+8^n)^{\frac{1}{n}} < 2^{\frac{1}{n}}(8^n)^{\frac{1}{n}}$$

$$\ell\, 8 \leqslant \ell\,(3^n+8^n)^{\frac{1}{n}} \leqslant \ell\, 8.2^{\frac{1}{n}}$$

Note: When taking limits < becom ≤

$$\therefore \quad 8 \leqslant \ell(3^n+8^n)^{\frac{1}{n}} \leqslant 8$$

$$\therefore \ell\,(3^n+8^n)^{\frac{1}{n}} = 8$$

* $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (will be proven in next section)

Note: Given a series $\sum_{n=1}^{\infty} a_n$ of positive terms (i.e. $a_1 \geqslant 0$, $a_2 \geqslant 0 \ldots$)

$\Rightarrow S_n$ is increasing for $a_n \geqslant 0$, hence if S_n can be shown to be bounded from above then S_n is convergent.

10.3 The integral test (remember it does NOT apply to all infinite series)

- The integral test can be applied if the nth term a_n can be written equal to $f(x)$ where $f(x)$ is a decreasing function of x.

$\Rightarrow \int_1^{\infty} f(x)\,dx = \lim_{A \to +\infty} \int_1^{A} f(x)\,dx$ } called limit of partial integral.

- If the limit is a finite number $\Rightarrow S_n$ is bounded by this number
  $\therefore$ series is convergent.
- If the limit is infinite $\Rightarrow$ series diverges

Example 1:

$$\sum_{n=1}^{\infty} \frac{1}{n} \; ; \; a_n = \frac{1}{n}$$

- 3 conditions must be satisfied to apply the integral test:

a) $a_n \to 0$; b) a_n is positive for all n ; c) a_n is decreasing

Since all these conditions are satisfied, take $f(x) = \frac{1}{x}$ ($f(x) > 0$ and decreasing for $x \geq 1$)

Then:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{A\to+\infty} \int_1^{A} \frac{1}{x}\,dx = \lim_{A\to+\infty} \ln x \Big|_1^{A} = \lim_{A\to+\infty} (\ln A - \ln 1) = +\infty$$

Thus the integral diverges $\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (for $\lim_{n\to+\infty} S_n = +\infty$)

Example 2:

Investigate the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \; ; \; a_n = \frac{1}{n^2}$$ (all 3 conditions are met)

Take $f(x) = \frac{1}{x^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{A\to+\infty} \int_1^{A} \frac{1}{x^2}\,dx = \lim_{A\to+\infty} -\frac{1}{x} \Big|_1^{A} = \lim_{A\to+\infty} \left(-\frac{1}{A} + 1\right) = 0 + 1 = 1$$

So the integral converges $\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Example 3:

The integral test can also be applied to a general form called p-series which is represented as $\sum_{n=1}^{\infty} \frac{1}{n^p}$; where it will converge if $p > 1$ and diverge if $p \leq 1$. [7]

Example 4:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$ (3 conditions are met)

cannot start from $n = 1$, for $\ln 1 = 0$, and $\frac{1}{0}$ is undefined.

Take $f(x) = \frac{1}{x \ln x}$

$$\Rightarrow \int_2^{\infty} \frac{1}{x \ln x} = \lim_{A\to+\infty} \int_2^{A} \frac{1}{x \ln x} = \lim_{A\to+\infty} \ln(\ln x) \Big|_2^{A}$$

$$= \lim_{A\to+\infty} [\ln(\ln A) - \ln(\ln 2)] = +\infty \; \therefore \text{series diverges}$$

*Proof of the integral test [8]

Lecture 5 (09SEP2016)

Problems to think about:

1) Given $\sum_{n=1}^{\infty} a_n$; $a_n \geq 0$, S_n its partial sum, suppose $\lim a_n S_n = 7$

a) Does $\sum_{n=1}^{\infty} a_n$ converge or diverge? Justify your answer.

b) Find $\lim a_n$ if it exists.

2) Find $\lim_{n\to+\infty} \dfrac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}}$ using S.T

3) Find infinite series in powers of x whose sum is equal to:

$$\frac{1}{x^2-3x-4} \text{ for all } x, \text{ where } |x|<1$$

[29OCT2016] My Solutions

1) Since $a_n \geq 0$ then $S_n \geq 0$, this means either S_n is bounded from above thus converging to a limit, or grows without bound, that is, diverge
Now suppose $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$. And also if $\sum_{n=1}^{\infty} a_n$ converges, this means that S_n is bounded and has limit L.

$\therefore \lim a_n S_n = 7 \Rightarrow 0 \cdot L = 7$
However this equality is impossible, thus $\sum_{n=1}^{\infty} a_n$ is diverging.
(S_n tends to infinite)

b) Since $\lim S_n = +\infty \quad \therefore \lim a_n = \frac{7}{\infty} = 0$

2) Let $f(x) = \frac{1}{\sqrt{x}}$ (where $f(x) > 0$ for all $x \geq 1$, and $\lim f(x) = 0$)
Then:

$$\int_1^{n+1} f(x)\,dx < f(1)+f(2)+f(3)+\dots+f(n) < f(1)+\int_1^{n} f(x)\,dx$$

$$\int_1^{n+1} x^{-\frac{1}{2}}\,dx < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}} < 1+\int_1^{n} x^{-1/2}\,dx$$

$$2\sqrt{x}\Big|_1^{n+1} < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}} < 1+2\sqrt{x}\Big|_1^{n}$$

$$2\sqrt{n+1}-2 < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}} < 1+2\sqrt{n}-2$$

Divide by $\sqrt{n}$

$$2\sqrt{\frac{n+1}{n}}-\frac{2}{\sqrt{n}} < \frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}} < 2-\frac{1}{\sqrt{n}}$$

Take limits:

$$2 < \frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}} < 2$$

$$\therefore \lim \frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\frac{1}{\sqrt{n}}}{\sqrt{n}} = 2$$

3) $\frac{1}{x^2-3x-4} = \frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$

let $x=-1$:

$0+3B=1$

$B=\frac{1}{3}$

let $x=4$

$-3A=1$

$A=\frac{1}{3}$

$\Rightarrow \frac{1}{x^2-3x-4} = \frac{\frac{1}{3}}{x+1} - \frac{\frac{1}{3}}{x-4} = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n x^n - \frac{1}{3}\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)\frac{x^n}{4^n}$

$= \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{3} + \sum_{n=1}^{\infty}\frac{1}{4^{n+1}}\frac{x^n}{3}$

10.3 If $f(x)$ is positive and decreasing on $1 \le x \le +\infty$, then we have two inequalities for their partial sum:

$$\int_1^{n+1} f(x)\,dx < f(1)+f(2)+f(3)+\dots+f(n) < f(1)+\int_1^{n} f(x)\,dx$$

Example: Investigate $\lim \frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{\ln n}$

Answer: Take $f(x)=\frac{1}{x}$

$$\int_1^{n+1}\frac{1}{x}dx < 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} < 1+\int_1^{n}\frac{1}{x}dx$$

$$\ln(n+1) < 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} < 1+\ln n$$

Divide by $\ln n$

$$\frac{\ln(n+1)}{\ln n} < \frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{\ln n} < \frac{\ln n+1}{\ln n}$$

Take limits:

$$1 \le \frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{\ln n} \le 1$$

$$\therefore \lim \frac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{\ln n} = 1$$

10.4 Comparison Test (Inequalities) Note: only works for series with positive terms.

- Simple Inequalities:

$n^2 < n^2+1$; $\frac{1}{n^2} > \frac{1}{n^2+1}$; $n^2 < 3n^2$; $\sin^2 n \le 1$; $\frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$

★ Direct Comparison Test (DCT)

- If $0 < a_n \leq b_n$ for all n from some point, then if $\sum_{n=1}^{\infty} b_n$ is convergent so is $\sum_{n=1}^{\infty} a_n$.

Proof:

Since we are given $0 < a_n \leq b_n$, then:

$$\left.\begin{aligned} a_1 &\leq b_1 \\ a_2 &\leq b_2 \\ a_3 &\leq b_3 \\ &\vdots \\ a_n &\leq b_n \\ &\vdots \end{aligned}\right\} \text{add all terms}$$

$$\underbrace{a_1 + a_2 + a_3 + \dots a_n \dots}_{\sum_{n=1}^{\infty} a_n} \leq \underbrace{b_1 + b_2 + b_3 + \dots + b_n \dots}_{\sum_{n=1}^{\infty} b_n}$$

Thus if $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ being smaller than it will also converge. (Both a_n and b_n are greater or equal to zero and increasing converging to a limit L)

Example 1:

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}+10}$$

- $n\sqrt{n}+10 > n\sqrt{n} \Rightarrow \frac{1}{n\sqrt{n}+10} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$, but $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is converging being a p-series with $p = 3/2 > 1$)

$\therefore \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}+10}$ is converging by DCT.

Example 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.01}+\sqrt{n}}$$

- $n^{1.01}+\sqrt{n} > n^{1.01} \Rightarrow \frac{1}{n^{1.01}+\sqrt{n}} < \frac{1}{n^{1.01}}$, but $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ is converging with $p = 1.01 > 1$ $\therefore \sum_{n=1}^{\infty} \frac{1}{n^{1.01}+\sqrt{n}}$ is also converging by DCT

* Limit Comparison Test (LCT)

Given $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$, we find a series $\sum_{n=1}^{\infty} b_n$, $b_n \geq 0$.

Then look at $\rho = \lim \frac{a_n}{b_n} = L$. If:

a) $0 < L < +\infty$ (fixed positive value), then both series behave alike. Which means if $\sum_{n=1}^{\infty} b_n$ is convergent so is $\sum_{n=1}^{\infty} a_n$, and if $\sum_{n=1}^{\infty} b_n$ is divergent so is $\sum_{n=1}^{\infty} a_n$.

b) $L = 0$, and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

c) $L = +\infty$, and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example 1:

$$\sum_{n=1}^{\infty} \frac{1}{6n^2 - n}$$; compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\rho = \lim \frac{\frac{1}{6n^2 - n}}{\frac{1}{n^2}} = \lim \frac{n^2}{6n^2 - n} = \frac{1}{6}$$; But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is converging ($p > 1$)

$\therefore \sum_{n=1}^{\infty} \frac{1}{6n^2 - n}$ also converges.

[30OCT2016]

Example 2:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$$; compare with $\sum_{n=1}^{\infty} \frac{1}{n^{1.4}}$

$$\rho = \lim \frac{\frac{\ln n}{n^{1.5}}}{\frac{1}{n^{1.4}}} = \lim \frac{\ln n}{n^{0.1}} = 0$$, but $\sum_{n=1}^{\infty} \frac{1}{n^{1.4}}$ is convergent $\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$ is also convergent by LCT.

(16SEP2016) Lecture 6

Ratio test & Root Test (+ absolute and conditional convergence)

Given $\sum_{n=1}^{\infty} a_n$ $a_n \geq 0$ (nonnegative), compute $\rho = \lim \frac{a_{n+1}}{a_n}$

If
- $0 \leq \rho < 1$ $\therefore$ series converges
- $\rho > 1$ $\therefore$ diverges
- $\rho = 1$ $\therefore$ inconclusive

Example 1: $\sum_{n=1}^{\infty} \frac{n}{2^n}$, find ρ

$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim \frac{n}{n+1} \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by ratio test.

Example 2: $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

$\rho = \lim \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} = 0 < 1 \therefore \sum_{n=1}^{\infty} \frac{5^n}{5!}$ converges by ratio test.

- Root test

Given $\sum_{n=1}^{\infty} a_n \quad (a_n \geq 0)$, compute $\rho = \lim \sqrt[n]{a_n} = \lim (a_n)^{\frac{1}{n}}$

if $0 \leq \rho \leq 1 \therefore$ series converges

$\rho > 1 \therefore$ series diverges

$\rho = 1 \therefore$ inconclusive

* Idea of the proof:

If $\lim (a_n)^{\frac{1}{n}} = \rho$

then $(a_n)^{\frac{1}{n}} \simeq \rho$ (for large n)

$\Rightarrow a_n \simeq \rho^n$ ← behaves like geometric series.

Example: $\sum_{n=1}^{\infty} \frac{5^n}{n}$

$\rho = \lim \sqrt[n]{a_n} = \lim \frac{5}{\sqrt[n]{n}} = 5 > 1 \therefore$ diverges.

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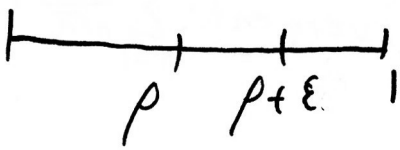
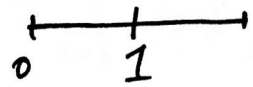
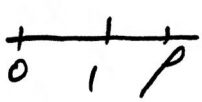


$\frac{a_{n+1}}{a_n} < \rho + \varepsilon$



$1 - \varepsilon < \frac{a_{n+1}}{a_n} < 1 + \varepsilon$



$\rho - \varepsilon < \frac{a_{n+1}}{a_n} < \rho + \varepsilon$

10.6 Test for alternating series (series with positive and negative terms)

eg. $\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$

* Reminder on Absolute Value

- Let x be a number $\begin{cases} |x| = x \text{ if } x \geq 0 \\ |x| = -x \text{ if } x < 0 \end{cases}$

* $|xy| = |x||y|$ → $x \leq |x|$, $-x \leq |x|$

* $|x+y| \leq |x|+|y|$ ∵ $|x| \pm x \geq 0$

* $|-x| = |x|$

* $|x-y| = |x+(-y)| \leq |x|+|y|$

- If $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n$ is <u>absolutely</u> convergent.

Example:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$$

Look at $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}}$, but $|\sin n| \leq 1$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}}_{\text{p series with } p = 3/2 > 1}$$

$\therefore \sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$ converges absolutely

- If $\sum_{n=1}^{\infty} a_n$ is given and $\sum_{n=1}^{\infty} |a_n|$ is <u>not</u> convergent, it is still possible that $\sum_{n=1}^{\infty} a_n$ converges, then it is conditionally convergent.

* Alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n \; ; \; u_n > 0$$

$$= u_1 - u_2 + u_3 - u_4 \cdots$$

If $u_1 \geq u_2 \geq u_3$

i) terms decrease in absolute value.

ii) $\lim_{n} u_n = 0$

iii) strictly alternating

then series is convergent.

Example:

a) $$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}$$

(i) $u_n = \frac{1}{n}$, and $\lim \frac{1}{n} = 0$

(ii) $\frac{1}{n}$ decreases

(iii) strictly alternating

} ∴ series converges by alternating series test (AST)

b) Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge absolutely?

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is diverging, ! $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent

- A trick for $\sum_{n=1}^{\infty} e^{\frac{1}{n}} - 1$; compare with $\sum \frac{1}{n}$ using LCT

$$\ell = \ell \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \ell_{x \to 0} \frac{e^{x} - 1}{x} = 1 \therefore \sum_{n=1}^{\infty} e^{\frac{1}{n}} - 1 \text{ diverges since } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

- Same for $\sum_{n=1}^{\infty} e^{\frac{\ln n}{n}} - 1 \Rightarrow$ compare with $\frac{\ln n}{n}$

Lecture 7 (19 SEP 2016)

Power Series

- Powers of x : $1, x, x^2, x^3 \ldots$
  Powers of $(x-a)$: $1, (x-a), (x-a)^2, (x-a)^3 \ldots$
- Power series : $\sum_{n=0}^{\infty} a_n (x-a)^n$
  - a_n ↓ coefficient of the power $(x-a)^n$
  - a ↳ center of power series (constant)

$$= a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots + a_n (x-a)^n + \cdots$$

Examples:

$$\sum_{n=0}^{\infty} x^n \quad , \quad \sum_{n=0}^{\infty} \frac{x^n}{n+1} \quad , \quad \sum_{n=0}^{\infty} \frac{\sin n}{2^n} (x-3)^n$$

[31 OCT 2016]

Quick Recap:
For geometric series we know it's n^{th} partial sum, $S_n = \frac{1 - x^{n+1}}{1-x}$ $(x \neq 1)$

- Given a power series :

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

- How to find all the values of x for which series is absolutely/conditionally convergent.

Answer: Apply ratio test in absolute value (:: we don't know if it's always positive)

i.e $$\ell \left| \frac{a_{n+1} (x-a)^{n+1}}{a_n (x-a)^n} \right| = \ell \left| \frac{a_{n+1}}{a_n} \right| |x-a|$$

- For absolute convergence we must have $\rho < 1$ (and solve for x)

Example 1:

Find all values of x for which series is convergent:

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

$$\rho = \ell \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \ell \, |x| \left| \frac{n+1}{n+2} \right| = |x| \quad \therefore \text{series is absolutely convergent for } |x|<1 \text{ and divergent for } |x|>1$$

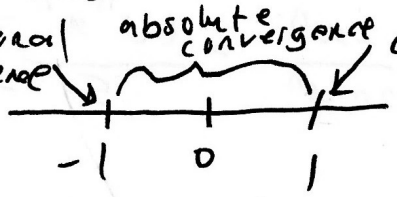
- Check end points:

* if $|x|=1$ then:

Case 1: $x=1$, $\sum_{n=0}^{\infty} \frac{1}{n+1}$ (diverges by integral test)

Case 2: $x=-1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ (converges by AST)

$\Rightarrow$



Example 2:

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2}$$

$\rho = |x+3| \quad \therefore$ series converges for $|x+3|<1$

$\Rightarrow \underbrace{-4 < x < -2}$ called interval of convergence

- Test end points:

At $x=-2$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (converges)

At $x=-4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (absolute convergence)

Example 3:

$$\sum_{n=0}^{\infty} \frac{2^n (2x-5)^{2n+3}}{n!}$$

$$\rho = \ell \frac{2^{n+1}}{\frac{(n+1)!}{2^n}} \cdot \frac{|2x-5|^{2n+5}}{|2x-5|^{2n+3}} = 0 < 1 \quad \therefore \text{series convergent for all } x$$

$\Rightarrow -\infty < x < +\infty$

- Next questions:
- How to find sum of a series (numerical/practical)
- Given the sum, how to find power series.

Example 1: What is the sum?

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$S_n = 1 - \frac{1}{n+1} \Rightarrow S = \lim S_n = 1$

Example 2:

$$\sum_{n=0}^{\infty} (5x)^n$$

$S = \frac{1}{1-5x}$; $|x| < \frac{1}{5}$

Example 3: Find the power series whose sum is $\frac{1}{x^2-3x-4}$

Answer: $\frac{1}{x^2-3x-4} = \frac{\frac{1}{5}}{x-4} - \frac{\frac{1}{5}}{x+1} = \frac{\frac{1}{5}}{-4(1-\frac{x}{4})} - \frac{\frac{1}{5}}{1-(-x)} = -\frac{1}{20}\left(\frac{1}{1-\frac{x}{4}}\right) - \frac{1}{5}\left(\frac{1}{1-(-x)}\right)$

$$= -\frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}\left[\frac{-x^n}{20 \cdot 4^n} - \frac{(-1)^n x^n}{5}\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{-1}{20 \cdot 4^n} + \frac{(-1)^{n+1}}{5}\right)x^n$$

* Properties:
Given $\sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$, they can be added, subtracted, multiplied, divided, differentiated and integrated.

- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$

(i) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

ii) $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$ (differentiated term by term)

iii) $\int_0^x f(t)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \int_0^x (a_n t^n)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

(21 SEP 2016) Lecture 8

Power series

Example:

a) $\frac{1}{2+3x} = \frac{1}{2(1+\frac{3x}{2})} = \frac{1}{2(1-(-\frac{3x}{2}))} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n 3^n}{2^{n+1}}}_{\text{coefficient}} x^n$

b) find power series of $\frac{1}{2+3x}$ about $a=1$ (center 1)

Answer:

$$\frac{1}{2+3x} = \frac{1}{2+3(x-1)+3} = \frac{1}{5(1+\frac{3}{5}(x-1))} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{3}{5}(x-1)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(3^n)}{5^{n+1}}(x-1)^n$$

10.8 Given f, point 0, find power series for f centered at a

The 3 steps

STEP 1	STEP 2	STEP 3
$f(x)$	$f(a)$	$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
$f'(x)$	$f'(a)$	
$f''(x)$	$f''(a)$	
$f'''(x)$	$f'''(a)$	
$\vdots$	$\vdots$	
$f^{n}(x)$	$f^{n}(a)$	

Example 1: $f(x) = e^x$, $a=0$

STEP 1	STEP 2
$f(x) = e^x$	$f(a) = 1$
$f'(x) = e^x$	$f'(a) = 1$
$f''(x) = e^x$	$f''(a) = 1$
$f'''(x) = e^x$	$f'''(a) = 1$
$\vdots$	$\vdots$
$f^{(n)}(x) = e^x$	$f^{(n)}(a) = 1$

STEP 3

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot x^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example 2: $f(x) = \ln(1+x)$, $a=0$

1

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{(-1)(-2)}{(1+x)^3}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

2

$\vdots$

$f^{(n)}(0) = (-1)^{n-1}(n-1)!$ for $n \geq 1$

3

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Second Method:

$* \frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x} \therefore \int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n = \ln(1+x)$

Detailed solution:

$$[\ln(1+t)]' = \frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\Rightarrow \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

Example 3: Find power series of xe^{-x^2} about $a=0$

Solution: Use $e^{(\ )} = \sum_{n=0}^{\infty} \frac{(\ )^n}{n!}$; so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$

$$\therefore xe^{-x^2} = \sum_{n=0}^{\infty} \frac{x(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$$

Extra: Find the 10th derivative of $f(x) = xe^{-x^2}$ at 0

Solution: $\frac{f^{(10)}(0)}{10!}$ = coefficient of x^{10} which is zero (since $2n+1=0$ $n \neq$ an integer)

$$-\frac{f^{(11)}(0)}{11!} = \frac{(-1)^5 x^{11}}{5!}$$ coefficient of x^{11}

Question: Why is it that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

To prove $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$, we must find $S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

and $\lim S_n$ must be $f(x)$.

- $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n - S_n(x) =$ error, If $\lim$ error $=0$, then full series $= f(x)$
- Suppose $\sum_{n=0}^{\infty} a_n(x-a)^n$ and partial sum. They are not equal but the difference between them is the error $= R_n(x)$, if $\lim R_n(x) = 0$, $f(x) =$ series.

Example:

$$S = \sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n > 0, \; a_n \text{ decreasing}, \; \lim a_n = 0$$

$$S_n = \sum_{k=1}^{n} (-1)^k a_k \; ; |\text{error}, (S - S_n)| \leq a_{n+1}$$