Math 202 Quiz II — May 10, 2012

Time: 70 minutes — *N. Nahlus & I. Itani*

Name

I.D .... (VERY CLEARLY)

Section Time..3:30.... (-2 points if incorrect)

1a) (8%) **Find Y**=Laplace (y) for the given IVP

$$y'' + 2y' + 3y = (t^{10} + e^{5t})^2 \quad ; \quad y(0) = 0 \;\&\; y'(0) = 4$$

DO not find y(t). Hint: Expand the right hand-**side**

Problem 1ab	15	15
Problem 2	5	3
Problem 3a,b	18	17
Problem 4ab	12	$11\frac{1}{2}$
Subtotal	50	$46\frac{1}{2}$
Problem 5	10	10
Problem 6	14	$12\frac{1}{2}$
Problem 7ab	14	14
Problem 8 ab	12	$11\frac{1}{2}$
Total over 100		94.5

$$s^2Y - s\,y(0) - y'(0) + 2sY - 2y(0) + 3Y = \mathcal{L}\left(t^{20} + 2t^{10}e^{5t} + e^{10t}\right)$$

$$s^2Y - 4 + 2sY + 3Y = \frac{20!}{s^{21}} + 2\frac{10!}{(s-5)^{11}} + \frac{1}{s-10}$$

$$Y(s^2+2s+3) = \frac{20!}{s^{21}} + \frac{(2)(10!)}{(s-5)^{11}} + \frac{1}{s-10} + 4$$

$$Y = \frac{\frac{20!}{s^{21}} + \frac{(2)(10!)}{(s-5)^{11}} + \frac{1}{s-10} + 4}{s^2 + 2s + 3}$$

1b) (7 %) **Find X**=Laplace (x(t)) and **Find Y**= Laplace (y(t)) in the system

$$y''+3x'+3x=0$$

$$y''+y'+2x=1+5t \quad ; \quad y(0)=-1,\ y'(0)=5, \quad x(0)=1,\ x'(0)=0.$$

Just get the 2 linear equations in X & Y; **then STOP!** **Do not simplify!**

Apply Laplace on both sides, of both equations.

$$\begin{cases} s^2Y - s\,y(0) - y'(0) + 3\left(sX - x(0)\right) + 3X = 0 \\ s^2Y - s\,y(0) - y'(0) + sY - y(0) + 2X = \frac{1}{s} + 5\frac{1}{s^2} \end{cases}$$

$$\begin{cases} s^2Y + s - 5 + 3sX - 3 + 3X = 0 \\ s^2Y + s - 5 + sY + 1 + 2X = \frac{1}{s} + \frac{5}{s^2} \end{cases}$$

$$\begin{cases} Y(s^2) + X(3s+3) = 8 - s \\ Y(s^2+s) + X(2) = \frac{1}{s} + \frac{5}{s^2} - s + 4 \end{cases}$$

2) (5 %) **Apply** Laplace Transform <u>**to change**</u> the following IVP <u>**to a separable DE; then STOP**</u>!

$$t\,y'' + y' + 5y = 0\ ;\quad y(0) = a\quad \&\ y'(0) = b$$

$\mathcal{L}(t\,f) = (-1)F'$

$(t\,y'') = (-1)\left(s^2Y - sy(0) - y(0)\right)'$

$\mathcal{L}\left(t\,y'' + y' + 5y\right) = 0$

$(-1)\left(s^2Y - sy(0) - y'(0)\right) + sY - y(0) + 5Y = 0$

$\boxed{\cancel{(-1)\,(s^2Y')}} + sY - a + 5Y = 0$

$-s^2Y' - a + Y(s+5) = 0$

$Y(s+5) - a = s^2\,\dfrac{dY}{dX}$

$dX = s^2 . dY\left(\dfrac{1}{Y(s+5) - a}\right)$

3a) (12 %) Find y(t) if its Laplace transform Y is

(i) $Y = \dfrac{1}{s^2(s^2+4)}$ by using Laplace **inverse** $\left(\dfrac{F(s)}{s}\right)$

(ii) $Y = \dfrac{s+8}{s^2+4s+29}$ (complete the square)

(i) 5

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \int_0^t\int_0^x \frac{\sin 2t}{2}\,dz\,dx$$

$$= \frac{1}{2}\int_0^t \left(-\frac{\cos 2t}{2}\right)\Big|_0^x dx$$

$$= \frac{1}{2}\int_0^t -\cos x + 1 \; dx$$

$$= \frac{1}{2}\left(-\sin x + x\Big|_0^t\right)$$

$$= \frac{1}{2}\left(-\sin t + t\right)$$

$$y(t) = \frac{1}{2}\left(t - \sin t\right)$$

(ii)

$$\mathcal{L}^{-1}\left(\frac{s+8}{s^2+4s+29}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s+8}{s^2+4s+4+25}\right) = \mathcal{L}^{-1}\left(\frac{s+8}{(s+2)^2+25}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{(s+2)+6}{(s+2)^2+25}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+25} + \frac{6}{(s+2)^2+25}\right)$$

$$= e^{-2t}\cos 5t + \frac{6}{5}e^{-2t}\sin 5t$$

3b) (6 %) Find y(t) if its Laplace transform Y is

$$F(s) = \left(\frac{(s+5)}{(s+5)^2+9}\right)^{(6)} \quad \text{(6}^{\text{th}} \text{ derivative)}$$

$$\mathcal{L}^{-1}\left\{\left(\frac{(s+5)}{(s+5)^2+9}\right)^{(6)}\right\}$$

$$= t^6 \cdot e^{-5t} \cdot \cos 3t$$

$$y(t) = t^6 \, e^{-5t} \cos 3t.$$

$$\mathcal{L}^{-1}\left(F^{(6)} \cdot (-1)^6\right) = t^6 \cdot f(t).$$

4a) (7 %) Find $f(t)$ in the equation

$$\int_0^t (f(\tau))^3 f(t-\tau)\, d\tau = \int_0^t \tau^8 (t-\tau)^{30}\, d\tau$$ (Hint: Take Laplace on both sides.)

$$\mathcal{L}\left(\int_0^t (f(\tau) \cdot f(t-\tau))\, d\tau\right) = \mathcal{L}\left(\int_0^t \tau^8 (t-\tau)^{30}\, d\tau\right)$$

$$\mathcal{L}\left(f(t) * f(t)\right) = \mathcal{L}\left(t^8 * t^{30}\right)$$

$$F(s) \cdot F(s) = \frac{8!}{s^9} \cdot \frac{30!}{s^{31}} = \frac{(8!)(30!)}{s^{40}}$$

$$F(s) = \frac{\sqrt{(8!)(30!)}}{s^{20}} \quad ; \quad \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{\sqrt{(8!)(30!)}}{s^{20}}\right)$$

$$f(t) = \sqrt{(8!)(30!)} \cdot \frac{t^{19}}{19!}$$

-1

4b) (5 %) Find $f(t)$ in the equation

$F(s) = \ln\left(\frac{s^2+1}{s^2}\right)$ **Hint:** Find the derivavtive $F'(s)$) AND $\ln\left(\frac{a}{b}\right) =$ ....

$$F'(s) = \frac{\dfrac{2s(s^2) - 2s(s^2+1)}{s^4}}{\dfrac{s^2+1}{s^2}} = \frac{\dfrac{-1}{s^4}}{\dfrac{s^2+1}{s^2}} = \frac{-1}{s^2(s^2+1)}.$$

$$t \cdot f(t) = \mathcal{L}^{-1}\left((-1)\frac{1}{s^2(s^2+1)}\right) = -\int_0^t\int_0^x \sin\tau \; d\tau\, dx$$

$$= -\int_0^t -\cos x + 1\, dx = -(-\sin t + t) = \sin t - t$$

$$f(t) = \frac{\sin t}{t} - 1.$$

5) (10 %) Determine the right **form** of y_p in the DE $y^{(4)} - y^{(2)} = 4x + x^2e^{-x}$
(**Do not find the constants** A, B, …..)

$y^{(4)} - y^{(2)} = 4x + x^2 e^{-x}$

$\cdot\ y^{(4)} - y^{(2)} = 0$ try $y = e^{mx}$.

$m^4 - m^2 = 0$

$m^2(m^2-1) = 0$

$m_1 = 0$ (double root)

$m_2 = 0$

or

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$, $m_3 = 1$

$m_4 = -1$

$$\boxed{\begin{array}{ll} y_1 = 1, & y_2 = x \\ y_3 = e^{x} & y_4 = e^{-x} \end{array}}$$

$y_p = (Ax + B) + (Cx^2 + Dx + E)\, e^{-x}$

$= (Ax^3 + Bx^2) + (Cx^3 + Dx^2 + Ex)\, e^{-x}.$

6) (14%) Solve the DE $y'' - y = \dfrac{2e^x}{e^x + e^{-x}}$

Note: The integrals are reasonable: multiply up & down by e^x) (Careful: we have zero y')

- $y'' - y = 0$

try $y = e^{mx}$, $m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m_1 = 1 \quad m_2 = -1$

$$\boxed{\begin{aligned} y_1 &= e^{x} \\ y_2 &= e^{-x} \end{aligned}}$$

Using variation of parameters:

$y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -(e^{-x})(e^{x}) - (e^{-x})(e^{x})$$
$$= -2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{-x} \\ \frac{2e^{x}}{e^{x}+e^{-x}} & -e^{-x} \end{vmatrix} = 0 - \left(\frac{2e^{x}}{e^{x}+e^{-x}}\right)(e^{-x})$$
$$= \frac{-2}{e^{x}+e^{-x}}$$

$-1\frac{1}{2}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{2e^{x}}{e^{x}+e^{-x}} \end{vmatrix} = \frac{2e^{2x}}{e^{x}+e^{-x}}$$

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{\frac{-2}{e^{x}-e^{-x}}}{-2}\,dx = \int \frac{1}{e^{x}+e^{-x}}\,dx = \int \frac{e^{x}}{e^{2x}+1}\,dx = ?$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{\frac{2e^{2x}}{e^{x}-e^{-x}}}{-2}\,dx = -\int \frac{e^{2x}}{e^{x}-e^{-x}}\,dx = -\int \frac{e^{3x}}{e^{2x}-1}\,dx = ?$$

$$y = c_1 e^{x} + c_2 e^{-x} + u_1 e^{x} + u_2 e^{-x}$$

7a) (7 %) Find the general solution of the Cauchy Euler DE $x^3 y^{(3)} + xy' - y = 0$ (Hint: m=1)

$x^3 y^{(3)} + xy' - y = 0$

try $y = x^m$

$m(m-1)(m-2) + m - 1 = 0$

$m(m^2 - 3m + 2) + m - 1 = 0$

$m^3 - 3m^2 + 2m + m - 1 = 0$

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)(m^2 - 2m + 1) = 0$

$(m-1)(m-1)^2 = 0$

$(m-1)^3 = 0$ $\quad m = 1$ (triple root).

$$\begin{array}{r} m^2 - 2m + 1 \\ m-1 \overline{)\, m^3 - 3m^2 + 3m - 1} \\ -(m^3 - m^2) \\ \hline -2m^2 + 3m - 1 \\ -(-2m^2 + 2m) \\ \hline m - 1 \\ -(m - 1) \\ \hline 0 \end{array}$$

$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 x + c_2 x \ln x + c_3 x \ln^2 x.$

7b) (7 %) Use the substitution $x = e^t$ to change the DE $x^2 y'' - 3xy' + y = 7e^{2x} + x^2 (\ln x)^3$

to a DE with constant coefficients. ***(THEN STOP). Do Not prove the Chain Rule formulas***

$t = \ln x$

$x = e^t$

$x y'_x = y'_t$

$x^2 y''_x = y''_t - y'_t$

$y''_t - y'_t - 3y'_t + y = 7e^{2e^t} + (e^{2t}) t^3$

$y''_t - 4y'_t + y = 7e^{2e^t} + (e^{2t}) t^3$

8a) (6 %) Use the existence/uniqueness Theorem to find all solutions of the BVP

$$y'''-xy''+y'=0 \quad ; \; y'(8)=8, \quad y''(8)=1, \quad y''(7)=1,$$

Hint: Try kx

let $u=y'$

$u''-xu'+u=0$

$u(8)=8$
$u'(8)=1$ IVP
$u'(7)=1$

Try $u=kx$
$u'=k$
$u''=0$

$0-xk+kx=0$
$0=0$

for $k=1$ $\quad u=x$

$u(8)=8$
$u'(8)=1$ $\quad + \quad u'(7)=1$
$u''(8)=0$

use uniqueness thm. before returning the hidden condition.

since $1, -x$ are continuous over 8

the eqn has good conditions

thus $\boxed{u=x}$ is the only soln for the IVP eqn.
$u''-xu'+u=0$

$y=\int u\,dx=\int x\,dx=\frac{x^2}{2}+c$

$\boxed{y=\frac{x^2}{2}+c}$ is the soln for the BVP eqn
$y'''-xy''+y'=0.$

8b) (6 %) In the Variation of parameters method we reach the point

$$(y_1u_1'+y_2u_2')' + p(x)(y_1u_1'+y_2u_2') + (y_1'u_1'+y_2'u_2') = f(x)$$

Suppose we make the assumption $y_1u_1'+y_2u_2' = x+5$ (INSTEAD of zero)

Find u_1' AND u_2'

let $y_1u_1' + y_2u_2' = x+5$

then we get $1 + p(x)(x+5) + (y_1'u_1' + y_2'u_2') = f(x)$.

$y_1'u_1' + y_2'u_2' = f(x) - 1 - p(x)(x+5)$

$$\Rightarrow \begin{cases} y_1u_1' + y_2u_2' = x+5 \\ y_1'u_1' + y_2'u_2' = f(x) - 1 - p(x)(x+5) \end{cases}$$

Using Cramer's Rule:

$$u_1' = \frac{\begin{vmatrix} x+5 & y_2 \\ f(x)-1-p(x)(x+5) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & x+5 \\ y_1' & f(x)-1-p(x)(x+5) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

- $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\,\mathcal{U}(t-a)$
- $\mathcal{L}\left(g(t)\,\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$

Laplace Transforms

$$\mathcal{L}(\cos Kt) = \frac{s}{s^2+K^2} \qquad \mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin Kt) = \frac{K}{s^2+K^2} \qquad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$ (Dirac)

$$\mathcal{L}(\mathcal{U}(t-a)) = \frac{e^{-as}}{s}$$ (step function)

$$\mathcal{L}\left(f(t-a)\,\mathcal{U}(t-a)\right) = e^{-as}F(s)$$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$$f * g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$ (Convolution)

$$\mathcal{L}(f * g) = F(s)\,G(s)$$

$$\mathcal{L}(y') = sY - y(0)$$

$$\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}$$

$$\mathcal{L}\left(g(t)\,\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s) \qquad n \geq 0$$

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)$$