Math 202 Quiz I

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Name

I.D .... (VERY CLEARLY)

Section number (or Thursday Time) ...3:30.p.m (-2 points if incorrect)

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0) ***Be Careful!*** There are many problems like: ***Change to .... Then STOP!***

1) **Change** the D.E $x^3y' = 7xy^2 + y^3 + 5x^2y\,e^{-(\frac{y}{x})^5}$ to separable then **STOP!**

Problem 1:	12%	12
Problem 2:	12%	12
Problem 3abc:	21%	21
Problem 4:	15%	15
Problem 5:	15%	15
Problem 6:	15%	14½
Problem 7:	10%	7½
Total over 100		97

$$x^3y' = 7xy^2 + y^3 + 5x^2y\,e^{-(\frac{y}{x})^5}$$

$(\div x^3)$

$$y' = \frac{7y^2}{x^2} + \frac{y^3}{x^3} + 5\frac{y}{x}\,e^{-(\frac{y}{x})^5}$$

This is a homogeneous D.E. of degree 3.

(let $u = \frac{y}{x}$, $dy = x\,du + u\,dx$)

$$\frac{dy}{dx} = 7\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 + 5\left(\frac{y}{x}\right) e^{-(\frac{y}{x})^5}$$

$$dy = \left(7u^2 + u^3 + 5u\,e^{-u^5}\right)dx$$

$$x\,du + u\,dx = \left(7u^2 + u^3 + 5u\,e^{-u^5}\right)dx$$

$$x\,du = \left(7u^2 + u^3 + 5u\,e^{-u^5} - u\right)dx$$

$$\frac{du}{7u^2 + u^3 + 5u\,e^{-u^5} - u} = \frac{dx}{x}$$; This is a separable D.E.

$$f(y)\,dy = f(x)\,dx.$$

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2) **Change** the D.E $a(x)y' - xy^{3/4} = \frac{x^2y}{x^4+1}$ to a **linear D.E** then **STOP!** $y' + p(x)y = f(x)$

$$a(x)y' - xy^{\frac{3}{4}} = \frac{x^2y}{x^4+1}$$

$$a(x)\,y' - \frac{x^2y}{x^4+1} = xy^{\frac{3}{4}}$$

$$y' - \frac{x^2}{(a(x))\,(x^4+1)}\cdot y = \frac{x}{a(x)}\cdot y^{\frac{3}{4}}$$

$\div y^{\frac{3}{4}}$

$$y^{-\frac{3}{4}}y' - \frac{x^2}{a(x)\cdot(x^4+1)}\cdot y y^{-\frac{3}{4}} = \frac{x}{a(x)}$$

let $u = yy^{-\frac{3}{4}} = y^{\frac{1}{4}}$

$u' = \frac{1}{4}y^{-\frac{3}{4}}y'$

$$4u' - \frac{x^2}{a(x)\cdot(x^4+1)}\cdot u = \frac{x}{a(x)}$$

$$u' - \frac{x^2}{a(x)\,(x^4+1)\cdot 4}\cdot u = \frac{x}{4a(x)}$$

This is linear in the form of $u' + p(x)\,u = f(x)$

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$y' + p(x)y = f(x)y^n$

3a) Consider the DE $xy+(y^2\sin(y)-x^2)y'=0$

Change the given D.E to a Bernoulli D.E in its standard form (showing n & p & f) then **STOP.**

$xy+(y^2\sin y - x^2)y'=0$, $-xy=(y^2\sin(y)-x^2)\frac{dy}{dx}$,

$-xy\frac{dx}{dy}=y^2\sin(y)-x^2$.

$\frac{dx}{dy}=-\frac{y}{x}\sin(y)+\frac{x}{y}$, $x'-\left(\frac{1}{y}\right)x=(-y\sin y)\cdot x^{-1}$

This is a Bernoulli D.E in the form $x'+p(y)x=f(y)x^n$.

where $p(y)=\frac{-1}{y}$ / $f(y)=-y\sin y$ / $x^n=x^{-1}$

3b) Solve the D.E $xy'-4y=x^6e^x$

$xy'-4y=x^6e^x$

$y'-4\frac{y}{x}=x^5e^x$; $y'+\left(\frac{-4}{x}\right)y=x^5e^x$ (This is a linear D.E $y'+p(x)y=f(x)$)

$I(x)=e^{\int p(x)dx}=e^{\int -\frac{4}{x}dx}=e^{-4\ln x}=x^{-4}$

$(I(x)\cdot y)'=I(x)\cdot f(x)$

$I(x)\cdot y=\int I(x)\cdot f(x)\,dx$

$x^{-4}y=\int x^{-4}\cdot x^5e^x\,dx=\int xe^x\,dx=xe^x-e^x+C$

let $u=x$ $dv=e^x dx$
$du=dx$ $v=e^x$

$\int xe^x dx = xe^x-\int e^x dx$
$=xe^x-e^x$

$y=x^4(xe^x-e^x+c)$

$$\boxed{y=x^5e^x-x^4e^x+x^4c}$$

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3c) **<u>Change</u>** the Riccati's DE $y' + xy^2 = 5x^2 y + 5$
to a Bernoulli DE then STOP. (<u>Hint</u>: Check $y = 5x$ is a solution of the D.E)

$y = 5x$
$y' = 5$

$5 + x(25x^2) = 5x^2(5x) + 5$

$5 + 25x^3 = 25x^3 + 5$ ✓ $y = 5x$ is a soln.

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let $u = y - 5x$

$y = u + 5x$

$\frac{dy}{dx} = \frac{du}{dx} + 5$

$y' = u' + 5$

$y' + xy^2 = 5x^2 y + 5$

$u' + \cancel{5} + x(u + 5x)^2 = 5x^2(u + 5x) + \cancel{5}.$

$u' + x(u^2 + 25x^2 + 10ux) = 5x^2 u + 25x^3.$

$u' + xu^2 + \cancel{25x^3} + 10ux^2 = 5x^2 u + \cancel{25x^3}.$

$u' + 5x^2 . u = -xu^2$

This is a Bernoulli D.E in the form.

$u' + p(x) u = f(x) . u^n$

where $p(x) = 5x^2$

$f(x) = -x$

$u^n = u^2$

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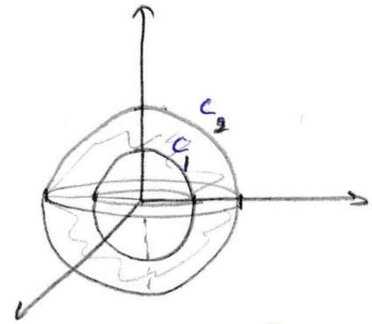
4) Use Gauss **Divergence Theorem** carefully to find the outward **flux** of the vector field

$$F = (\rho^{-5} + \rho^{-4} + \rho^{-3})(7x, 8y, 9z)$$

across the surface surrounding the region $1 \le \rho \le 2$ between spheres.
(Hint: Break the surface into two spheres)



$$\iint_{1\le\rho\le 2} F\cdot n\, d\sigma = \iint_{\rho=2} F\cdot n\, d\sigma - \iint_{\rho=1} F\cdot n\, d\sigma$$

• $$\iint_{\rho=2} F\cdot n\, d\sigma = \iiint \operatorname{div}(F)\cdot dV = \iiint \operatorname{Div}\left((\rho^{-5}+\rho^{-4}+\rho^{-3})(7x, 8y, 9z)\right) dV$$

by ½ subsitution $$\iiint \operatorname{Div}\left(\frac{1}{2^5}+\frac{1}{2^4}+\frac{1}{2^3}\right)(7x+8y+9z)\, dV$$

$$= \frac{7}{2^5}\cdot\iiint \operatorname{div}(7x+8y+9z)\, dV = \frac{7}{2^5}\cdot\iiint(7+8+9)\, dV = \frac{7}{2^5}\cdot 24\cdot V_{c_2}$$

$$= \frac{7}{2^5}\cdot 24\left(\frac{4}{3}\pi (2)^3\right) = 56\pi.$$

• $$\iint_{\rho=1} F\cdot n\, d\sigma = \iiint \operatorname{div}(F)\, dV \xrightarrow[\text{substitution}]{\text{by } 1/2} 3\iiint \operatorname{div}(7x+8y+9z)\, dV$$

$$= 3\cdot\iiint(7+8+9)\, dV = 3\cdot 24\cdot\left(\frac{4}{3}\pi (1)^3\right) = 96\pi.$$

$$\Rightarrow \iint_{1\le\rho\le 2} F\cdot n\, d\sigma = 56\pi - 96\pi = -40\pi.$$

5) Answer **ONE OF THE FOLLOWING. YES, choose one part!**

Choose i) Verify Stokes' Theorem (i.e, compute BOTH sides of the formula) for $F=(y^2, x, z^2)$ on the plane surface $y+z=7$ bounded by the cylinder $x^2+y^2=4$.

OR choose ii) Verify Stokes' Thm. for (i.e, compute BOTH sides of the formula) for
$F=(3ye^{3xyz}+xz^3\sin(y), 10x+5yz\cos(x), 5\sin(xy)+xye^{\sin xy})$
across the hemi-ellipsoid $x^2+4y^2+z^2=10$ and $z\geq 1$ whose boundary C is the ellipse $x^2+4y^2=9$ in the plane $z=1$. (Hint: Area (Ellipse of radii a, b) = πab)

Stoke's theorem: $\iint \text{Curl}(F)\cdot n \, d\sigma = \oint M\,dx + N\,dy + P\,dz.$

- $\oint_C M\,dx+N\,dy+P\,dz = \oint_C y^2\,dx + x\,dy + z^2\,dz = \oint_C y^2\,dx + x\,dy + (7-y)^2(-dy)$

$\begin{pmatrix} y+z=7 \\ z=7-y \\ dz=-dy \end{pmatrix}$ $= \oint \overset{M}{y^2}\,dx + \overset{N}{\left(x-(7-y)^2\right)}\,dy$

then by Green's theorem; $= \iint (\overset{N_x}{1} \overset{-M_y}{-2y})\,dA = \int_0^{2\pi}\int_0^2 1-2(r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{r^2}{2} - 2\frac{r^3}{3}\sin\theta \Big|_0^2 d\theta = \int_0^{2\pi} 2 - 2.\frac{2^3}{3}\sin\theta\,d\theta$

$= 2\left(\theta + \frac{2^3}{3}\cos\theta\right)\Big|_0^{2\pi} = 2\left(2\pi + \cancel{\frac{8}{3}(1)} - 0 - \cancel{\frac{8}{3}(1)}\right) = 4\pi.$

- $\text{Curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(1-2y)$

$= (0, 0, 1-2y).$

$\vec{n} = \pm\frac{\nabla g}{|\nabla g|} = \oplus\frac{(0,1,1)}{\sqrt{2}}$ $(g: y+z-7=0)$. choose $\oplus$.

$d\sigma = \frac{|\nabla g|\,dA}{|g_z|} = \frac{\sqrt{2}\,dA}{|1|} = \sqrt{2}\,dA$

continue on back of the sheet

$$\iint \operatorname{Curl}(F)\cdot n\, d\sigma = \iint (0,0,\ 1-2y)\,(0,1,1)\ dA$$

$$= \iint (1-2y)\ dA = \int_0^{2\pi}\!\!\int_0^{2} (1-2r\sin\theta)\, r\,dr\,d\theta = \ ---\ = 4\pi.$$

$$\therefore \iint \operatorname{Curl}(F)\cdot n\, d\sigma = \oint M\,dx + N\,dy + P\,dz.$$

$\Rightarrow$ Stoke's theorem is verified.

6a) Under what conditions, we have **Div** (f **grad** f)=**0** ? Justify your answer.
(Hint: Write **grad**(f)=(f_x, …, …..)

$\vec{\nabla} f = f_x + f_y + f_z$, $f . \vec{\nabla} f = f f_x + f f_y + f f_z$.

$\mathrm{Div}(f . \vec{\nabla} f) = f_x f_x + f f_{xx} + f_y f_y + f f_{yy} + f_z f_z + f f_{zz} = f_x^2 + f_y^2 + f_z^2 + f(f_{xx} + f_{yy} + f_{zz})$

if $\vec{\nabla} f = 0$

, so if f satisfy laplace's eqn.

where $f_{xx} + f_{yy} + f_{zz} = 0$

then $\mathrm{Div}(\vec{\nabla} f) = 0$.

no !

$\mathrm{div} f \nabla f = 0$ iff $f(\Delta f) = |\nabla f|^2$

6b) Let S be a semi-closed (solid) surface with surface Area 200. Suppose we have a vector field F=(M, N, P) such that $\| Curl(F) \| \leq 257$ & $\| F \| \leq 412$ over S

Use the **Stokes' theorem** to find an upper bound on $| \oint Mdx + Ndy + Pdz |$ over C
Where C is the boundary of the surface S

$-\frac{1}{2}$

$\iint_S \mathrm{Curl}(F) . n \, d\sigma = \oint_C M dx + N dy + P dz$.

the $\left| \iint_S \mathrm{Curl}(F) . n \, d\sigma \right| = \left| \oint M dx + N dy + P dz \right|$

$|\mathrm{Curl}(F) . n| = |\mathrm{Curl}(F)| . |n| . \cos\theta \leq 257$.

$\iint d\sigma = A = 200$.

$\iint \|\mathrm{curl}\, F\| \|n\| \leq$

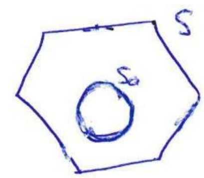
$\left| \iint_S \mathrm{Curl}(F) . n \, d\sigma \right| \leq 257 \times 200 = 51400$

$\left| \oint M dx + N dy + P dz \right| \leq 51400$.

7) Let S be a smooth polygonal surface around the origin with inner region D such that volume (D)=100. Find the outward flux of $F = (\rho^{-3})(x, y, z) + (x, y, z)$ across S. (<u>Hint:</u> Use <u>without proof</u> that $\text{Div}((\rho^{-3})(x, y, z)) = 0$ outside the origin.)

$-2\frac{1}{2}$

S_a : is a spher of center a.
radius



$$\iint_S F\cdot n\,d\sigma - \iint_{S_a} F\cdot n\,d\sigma = \iiint \text{div}(F)\,dV$$

(between)

$$\iint_S F\cdot n\,d\sigma = \iiint \text{div}\left(\cancel{\rho^{-3}(x,y,z)}\,dV\right) + \iiint \text{div}(x,y,z)\,dV + \iint_{S_a} F\cdot n\,d\sigma$$

$$= 3\cdot V_D + \iiint \text{div}\left(a^{-3}(x,y,z) + (x,y,z)\right) dV$$

we still can't use div. thm here ??

$$= 300 + \left(\frac{1}{a^3}+1\right)\iiint (x,y,z)\,dV$$

$$= 300 + \left(\frac{1}{a^3}+1\right)3\cdot\left(\frac{4}{3}\pi a^3\right)$$

$$= 300 + (1+a^3)(4\pi).$$