MATH 201: Calculus and Analytic Geometry III
Fall 2017-2018, Exam 1, Duration: 60 min.

Problem	1a	1b	1c	2a	2b	2c	2d	2e	3a	3b	4a	4b	Total
Points	8	8	8	8	8	8	8	8	16	10	5	5	100
Scores	8	8	8	8	7	8	8	8	16	10	5	1	95

Name:_ AUB ID:_

Please circle your section:

Section 1
MWF 3, Karam
Recitation F. 11

Section 2
MWF 3, Karam
Recitation F. 8

Section 3
MWF 3, Karam
Recitation F. 10

Section 4
MWF 3, Karam
Recitation F. 9

Section 5
MWF 10, Shayya
Recitation T. 11

Section 6
MWF 10, Shayya
Recitation T. 12:30

Section 7
MWF 10, Shayya
Recitation T. 2

Section 8
MWF 10, Shayya
Recitation T. 5

Section 9
MWF 11, Yamani
Recitation F. 3

Section 10
MWF 11, Yamani
Recitation F. 4

Section 12
MWF 2, Nahlus
Recitation M. 8

Section 13
MWF 2, Nahlus
Recitation M. 9

Section 16
MWF 9, Mourtada
Recitation Th. 9:30

Section 17
MWF 9, Mourtada
Recitation Th. 2

Section 19
MWF 1, Nahlus
Recitation F. 10

Section 20
MWF 1, Nahlus
Recitation F. 8

Section 22
MWF 10, Abi Khuzam
Recitation F. 4

Section 23
MWF 10, Abi Khuzam
Recitation F. 2

Section 26
MWF 11, Aoun
Recitation Th. 11

Section 27
MWF 11, Aoun
Recitation Th. 12:30

Section 28
MWF 11, Aoun
Recitation Th. 5

Notes before solving the exam:

1) You have to solve the recommended problems in the book after understanding each chapter from the book and the notes.

2) Please understand that this exam is solved by students, and it may contain some mistakes.

3) If you have any questions or concerns, let us know through our mail: insightclub@gmail.com.

GOOD LUCK :)

INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.
(b) Closed book. No notes. No calculators. No cellphones.
(c) UNLESS CLEARLY SPECIFIED OTHERWISE, THE BACKSIDE OF THE PAGES WILL NOT BE GRADED,

Problem 1

(8 pts each)Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = n(e^{1/n} - 1)$

$$\lim a_n = \lim \frac{e^{1/n} - 1}{\frac{1}{n}} = \lim \frac{\frac{-1}{n^2} e^{1/n}}{\frac{-1}{n^2}} \left(\begin{array}{c}\text{by} \\ \text{H.R}\end{array}\right)$$

$$= \lim e^{1/n} = e^0 = 1$$

then a_n converges

8

(b) $b_n = \dfrac{7^n . \sin(e^n)}{5^{-n} . n!}$

$$-1 \leq \sin(e^n) \leq 1$$

then $\dfrac{-7^n}{5^{-n} . n!} \leq \dfrac{7^n \sin(e^n)}{5^{-n}} \leq \dfrac{7^n}{5^{-n} . n!}$

but $\lim \dfrac{7^n}{5^{-n} . n!} = \lim \dfrac{35^n}{n!} = 0$ (Basic Limit)

and $\lim \dfrac{-7^n}{5^{-n} . n!} = \lim \dfrac{-35^n}{n!} = 0$

$\Rightarrow$ by sandwich theorem $\lim b_n = 0$

then b_n converges

8

(c) $c_n = \left(\frac{-n}{n+1}\right)^n$

$$\lim c_n = \lim \frac{n^n (-1)^n}{n^n (1+\frac{1}{n})^n} = \lim \frac{(-1)^n}{(1+\frac{1}{n})^n}$$

• if n is even $(n = 2k)$

$$\lim c_n = \lim \frac{(-1)^{2k}}{(1+\frac{1}{2k})^{2k}} = \frac{1}{e}$$ (Basic limit $\lim (1+\frac{1}{n})^n = e$

• if n is odd $(n = 2k+1)$

8

$$\lim c_n = \lim \frac{(-1)^{2k+1}}{(1+\frac{1}{2k+1})^{2k+1}} = \frac{-1}{e}$$

since the 2 limits are different then c_n diverges

Problem 2

(8 pts each) Which of the following series converge, and which diverge?

Find the sum of the series when possible.

(a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{5^n}+\frac{3^n}{5^{n+1}}\right)$

8

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \frac{3^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}}$$

• $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n$ which is a geo. series of $r = \frac{-1}{5}$ $\left(|r| = \left|\frac{-1}{5}\right| = \frac{1}{5} < 1\right)$ then $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$ converges having sum $S_1 = \frac{1}{1-\left(\frac{-1}{5}\right)} = \frac{5}{6}$

• $\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ which is a geo. series of $\left(|r| = \left|\frac{3}{5}\right| < 1\right)$ $r = \frac{3}{5}$ then $\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}}$ converges having sum

$$S_2 = \left(\frac{1}{5}\right)\frac{1}{1-\frac{3}{5}} = \frac{5}{2}\left(\frac{1}{5}\right) = \frac{1}{2}$$

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \frac{3^n}{5^{n+1}}$ converges (sum of 2 convergent series)

having sum $S = S_1 + S_2 = \frac{5}{6} + \frac{1}{2} = \frac{8}{6} = \frac{4}{3}$

(b) $\sum_{n=2}^{\infty}\left(\frac{1}{n\ln^2 n}\right)$ compare ~~$\sum_{n=2}^{\infty}\left(\frac{1}{n\ln^2 n}\right)$ with $\sum_{n=0}^{\infty} \frac{1}{n^{1.1}}$~~

by L. C. T.

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~~$\lim \frac{\frac{1}{n\ln^2 n}}{\frac{1}{\ln n}} = \lim \frac{1}{n\ln n} =$~~

~~$\lim \frac{\frac{1}{n\ln^2 n}}{\frac{1}{n^{1.1}}} = \lim \frac{n^{0.1}}{}$~~

7

let $f(x) = \frac{1}{x\ln^2 x}$ $\left(f(x) = f(n)\right)$

f is positive and cont.

$f'(x) = \frac{-(\ln^2 x + 2\ln(x))}{(x\ln^2(x))^2} < 0$ for $x \geq 2$ then f is also decreasing

then $\int_2^{\infty} f(x)\,dx$ and $\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$ both cv or both div

$\int_2^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_2^N \frac{dx}{x\ln^2 x}$

$\boxed{\begin{gathered}\text{for } \ln x = t \\ \frac{dx}{x} = dt \\ dx = t\,x\end{gathered}}$

then $= \lim_{N\to\infty} \int_2^N t^{-2}\,dt = \lim_{N\to\infty} \frac{t^{-3}}{-3}\Big|_2^N$

-1

$-\frac{1}{\ln x}\Big]_2^{\infty}$

$= \lim_{N\to\infty} \frac{1}{-3\ln^3(x)}\Big|_2^N = \lim_{N\to\infty} \frac{1}{-3\ln^3(N)} + \frac{1}{3\ln^3(2)} = \frac{1}{3\ln^3(2)}$

$\frac{1}{\ln 2}$

then $\int_2^\infty f(x)\,dx$ converges

then $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$ also converges by integral test

(c) $\sum_{n=1}^{\infty}\left(\frac{4n+5}{7n-4}\right)^{n}$

~~$\lim \left(\frac{4n+5}{7n-4}\right)^n = \lim \frac{4n^n}{}$~~

by root test: $\lim \sqrt[n]{\left(\frac{4n+5}{7n-4}\right)^n} = \lim \frac{4n+5}{7n-4} = \frac{4}{7} < 1$

then $\sum_{n=1}^{\infty}\left(\frac{4n+5}{7n-4}\right)^n$ converges by root test

8

(d) $\sum_{n=2}^{\infty}\left(\frac{\sqrt{\ln n}+\ln n}{n+n^{1.5}}\right)$

by doing L.C.T with $\sum_{n=2}^{\infty}\frac{\ln n}{n^{1.5}}$

$$\lim \frac{\frac{\sqrt{\ln n}+\ln n}{n+n^{1.5}}}{\frac{\ln(n)}{n^{1.5}}} = \lim \frac{\frac{\ln(n)\left(\cancel{\frac{\sqrt{\ln(n)}}{\ln(n)}}^{\to 0}+1\right)}{n^{1.5}\left(\cancel{\frac{n}{n^{1.5}}}^{\to 0}+1\right)}}{\frac{\ln(n)}{n^{1.5}}}$$

$$= \lim \frac{\ln(n)\cdot n^{1.5}}{\ln(n)\cdot n^{1.5}} = 1$$

but if we do L.C.T for $\sum_{n=2}^{\infty}\frac{\ln n}{n^{1.5}}$ with $\sum_{n=2}^{\infty}\frac{1}{n^{1.1}}$

we get: $\lim \frac{\frac{\ln(n)}{n^{1.5}}}{\frac{1}{n^{1.1}}} = \lim \frac{\ln(n)}{n^{0.4}} = 0$ (growth of $n >\!\!>\!\!>$ growth of $\ln(n)$)

then $\sum_{n=2}^{\infty}\frac{\ln n}{n^{1.5}}$ cv by L.C.T since $\sum_{n=2}^{\infty}\frac{1}{n^{1.1}}$ cv

by p-series $(1.1>1)$ $\Rightarrow \sum_{n=2}^{\infty}\frac{\sqrt{\ln n}+\ln(n)}{n+n^{1.5}}$ cv

by L.C.T.

(e) $\sum_{n=1}^{\infty} (arctan(n+1) - arctan(n))$ (**Note:** $arctan(n) = tan^{-1}(n)$)

$$\sum_{n=1}^{\infty} arctan(n+1) - arctan(n) = (\cancel{arctan 2} - arctan 1) + (\cancel{arctan 3} - \cancel{arctan 2})$$

$$+ (arctan 4 - \cancel{arctan 3}) + \cdots + (arctan(n+1) - arctan n)$$

then by telescoping series; (S_n is the n^{th} partial sum)

$S_n = arctan(n+1) - arctan 1 = arctan(n+1) - \frac{\pi}{4}$

but $\lim_{n \to \infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

then $\sum_{n=1}^{\infty} arctan(n+1) - arctan(n)$ converges by Telescoping series having sum $= \frac{\pi}{4}$

Problem 3

(**a**)(16 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)2^n}$$

check back of page!!

(Remember to check the endpoints.)

by ratio test: $\lim \left| \frac{(x-3)^{n+1}}{(n+2)(2^{n+1})} \cdot \frac{(n+1)(2^n)}{(x-3)^n} \right|$

$= \lim \frac{|x-3|^n |x-3|}{|x-3|^n} \cdot \frac{(n+1)}{n+2} \cdot \frac{2^n}{2^n \cdot 2}$

$= |x-3| \cdot 1 \cdot \frac{1}{2} = \frac{|x-3|}{2} < 1$ for the power series to converge then: $|x-3| < 2$

$-2 < x-3 < 2$

$1 < x < 5$

for $x=1$ $\sum_{n=1}^{\infty} \frac{(-1)^n (2)^n}{(n+1)(2)^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ let $a_n = \frac{(-1)^n}{n+1} = (-1)^n c_n$

Applying AST: $c_n = \frac{1}{n+1} > 0$

$\lim c_n = \lim \frac{1}{n+1} = 0$

$f(x) = \frac{1}{x+1}$ ($f(x) = f(n)$ for c_n) then $f'(x) = \frac{-1}{(x+1)^2} < 0$ for $x \geq 1$

then f is decreasing and thus c_n is decreasing

then the 3 conditions for AST are satisfied

($\lim c_n = 0$ and $c_n \searrow$ and $c_n > 0$)

then for $x = 1$ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)2^n}$ is convergent (conditionally)

for $x = 5$ $\sum_{n=0}^{\infty} \frac{2^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

~~(but $n+1 > n$ then $\frac{1}{n+1} > \frac{1}{n}$)~~

by doing L.C.T with $\sum_{n=0}^{\infty} \frac{1}{n}$ we get:

$$\lim \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim \frac{n}{n+1} = 1$$

but since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \leq 1$) then

$\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges by L.C.T then

for $x = 5$ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)2^n}$ diverges

$\Rightarrow$ the interval of convergence is:

$$1 \leq x < 5$$

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(b) (10 pts) For $x = 2.8$, use the alternating series estimation theorem (ASET) to approximate the sum of the series in part **(a)** with an error of magnitude no greater than 10^{-4}. (Make sure to justify why the conditions for ASET are satisfied.)

for $x = 2.8$ $\quad \sum_{n=1}^{\infty} \frac{(-0.2)^n}{(n+1)(2^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n (0.1)^n}{(n+1)}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)10^n}$ then: let $c_n = \frac{1}{(n+1)10^n}$

checking conditions for ASET:

- $\lim c_n = \lim \frac{1}{(n+1)10^n} = 0$
- $c_n = \frac{1}{(n+1)10^n} > 0$

- if $f(x) = \frac{1}{(x+1)10^x}$ $\left(\begin{matrix} f(x) = f(n) \\ \text{for } c_n \end{matrix} \right)$

$f'(x) = \frac{-(10^x + (x+1)\ln 10 \cdot 10^x)}{[(x+1)(10^x)]^2} < 0$ for $x \geq 1$

then f is decreasing and thus $\cancel{c_n}$ c_n is decreasing also.

then the conditions for ASET are satisfied.

S $|\text{error}| = |L - S_n| < \left|\begin{matrix}\text{first unused} \\ \text{term}\end{matrix}\right| < 10^{-4}$

$S = \frac{-1}{20} + \frac{1}{300} - \frac{1}{4000} + \frac{1}{50000}$

each of these terms has a magnitude greater than 10^{-4}

but $\left|\frac{1}{4000}\right| = \frac{1}{4} 10^{-4} < 10^{-4}$

then $\frac{1}{5000}$ is the first neglected term

$\Rightarrow S_n = \frac{-1}{20} + \frac{1}{300} - \frac{1}{4000} = \frac{-20}{4000} +$

$$S = \underbrace{\frac{-1}{20} + \frac{1}{300} - \frac{1}{4000}}_{\text{each of these terms has a magnitude greater than } 10^{-4}} + \frac{1}{50000}$$

but $\left|\frac{1}{50000}\right| = \frac{1}{5} \times 10^{-4} < 10^{-4}$ then 5×10^{-4} is the first neglected term and S is appoximated by the first 3 terms

$$S = \frac{-1}{20} + \frac{1}{300} - \frac{1}{4000} = \frac{-23}{12000}$$

Problem 4

Suppose that the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} c_n$ satisfy

- $\sum_{n=1}^{\infty} a_n$ converges
- $\sum_{n=1}^{\infty} c_n$ converges
- $a_n \le b_n \le c_n$ for all n.

(a) (5 pts) Prove that $b_n \to 0$. since both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$

converges then $\lim a_n = \lim c_n = 0$

but $a_n \leq b_n \leq c_n$ then by sandwich theorem

$\lim b_n = 0$ (~~since~~ $\lim b_n = \lim a_n = \lim c_n = 0$)
sandwich theorem

5

(b) (5 pts) Prove that $\sum_{n=1}^{\infty} b_n$ converges.

since ~~$b_n \geq a_n$~~ $b_n \leq c_n$

1

and $\sum_{n=1}^{\infty} c_n$ converges then by D.C.T

$\sum_{n=1}^{\infty} b_n$ also converges