Math 201–Exam 2 (Fall 15) Prof. B. Shayya and Dr. H. Yamani

- Please write your **section number** on your booklet.
- Please place your **student ID card** on the desk in front of you.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- **Unjustified** answers will receive little or no credit.

Problem 1 (answer on pages 1 and 2 of the booklet.)

(a) (8 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x\,y^4}{3x^8+y^4}$$

exist? Why or why not?

(b) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^6\,y}{3x^8+y^4}?$$

(c) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x\,y}{3x+y}?$$

Problem 2 (answer on page 3 of the booklet.)

Suppose $f(x,y,z)$ is a differentiable function of three variables such that

$$f(1,2,3)=f(1,1,-4)=4, \qquad \nabla f(1,1,-4)=\mathbf{i}+\mathbf{j}+\mathbf{k}, \qquad \nabla f(8,3,-4)=3\mathbf{i}-\mathbf{j}+\mathbf{k}.$$

Let

$$x=5r+3s, \qquad y=2r+s, \qquad z=-2(r^2+s^2), \quad \text{and} \quad w=f(x,y,z).$$

(i) (12 pts) Find $\partial w/\partial r$ and $\partial w/\partial s$ at the point $(r,s)=(1,1)$.

(ii) (12 pts) Estimate $f(1.01, 1.02, -3.98)$.

Problem 3 (answer on page 4 of the booklet.)

Consider the function $f(x)=e^x$.

(a) (12 pts) Use Taylor's theorem to find a power series expansion for $f(x)$ about the point $x=0$.

(b) (12 pts) Find the Taylor polynomial $p_3(x)$ generated by f at $x=0$. Then use Taylor's theorem to estimate the error resulting from the approximation $e^{-0.1}\approx p_3(-0.1)$. Conclude that $e^{0.1}<10/9$.

Problem 4 (answer on pages 5 and 6 of the booklet.)

Let C be the two-sided cone $z^2=x^2+y^2$.

(i) (15 pts) Find the tangent plane and normal line of C at the point $(3,4,5)$.

(ii) (10 pts) Let S be the set of all points (x,y,z) in C such that the normal line of C at (x,y,z) is perpendicular to the vector $\mathbf{v}=\mathbf{i}+\mathbf{j}+\sqrt{2}\,\mathbf{k}$. Prove that S lies on a line L.

(iii) (3 pts) Find parametric equations for the line L from part (ii).

1) a) $\left|\dfrac{xy^4}{3x^8+y^4}\right| \leq \dfrac{\sqrt{x^2+y^2}\,(3x^8+y^4)}{3x^8+y^4} = \sqrt{x^2+y^2} \to 0$ as $(x,y)\to(0,0)$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{xy^4}{3x^8+y^4} = 0$ (exists)

b) $\lim\limits_{\substack{x\to 0\\ y=mx^2}} \dfrac{x^6y}{3x^8+y^4} = \lim\limits_{x\to 0} \dfrac{mx^8}{3x^8+m^4x^8}$

$= \lim\limits_{x\to 0} \dfrac{x^8 \times m}{x^8(3+m^4)} = \lim\limits_{x\to 0} \dfrac{m}{3+m^4} = \dfrac{m}{3+m^4}$

which depends on m so if $m=1$ $L=\frac{1}{4}$

if $m=-1$ $L=\frac{-1}{4}$

$\therefore$ the limit doesn't exist.

c) We need a function $f(x)$ s.t. $y=-3x$ is tangent to $y=f(x)$ at $x=0$

Using Taylor polynomials (theorem) $P_1(x)=-3x$

By Taylor's theorem:

$f(x) = P_1(x) + R_1(x)$

$= -3x + \dfrac{f''(c)}{2!}(x)^2 = -3x + mx^2$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{3x+y} = \lim\limits_{\substack{x\to 0\\ y=-3x+mx^2}} \dfrac{x(-3x+mx^2)}{3x-3x+mx^2} = \lim\limits_{x\to 0} \dfrac{-3+\cancel{mx}^{\,0}}{m}$

$= \dfrac{-3}{m}$

which depends on m, Doesn't exist.

2) i) $\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$

$(r,s)=(1,1)$

$\Rightarrow (x,y,z) = (8,3,-4)$

$= \nabla f(8,3,-4)\cdot\left(\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) \quad (r,s)=(1,1)$

$= (3i-j+k)\cdot(5,2,-4r) \quad (r,s)=(1,1)$

$= (3i-j+k)\cdot(5,2,-4)$

$= (3,-1,1)\cdot(5,2,-4) = 15-2-4 = 9$

$\frac{\partial w}{\partial s} = \nabla f(8,3,-4)\cdot\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right) \quad (r,s)=(1,1)$

$= (3,-1,1)\cdot(3,1,-4s)$ //

$= (3,-1,1)\cdot(3,1,-4)$

$= 9-1-4 = 4$

ii) $\Delta f \simeq (\nabla f(1,1,-4)\cdot\vec{u}) \times ds$

$\vec{u} = \frac{(1.01, 1.02, -3.98) - (1,1,4)}{\text{magnitude}} = \frac{(0.01, 0.02, 0.02)}{\sqrt{0.01^2+0.02^2+0.02^2}}$

$= \frac{1}{3}(1,2,2)$

$\nabla f(1,1,-4)\cdot\vec{u} = \frac{1}{3}(1,1,1)\cdot(1,2,2) = \frac{5}{3}$

$\Delta f \simeq \frac{5}{3}\times ds = \frac{5}{3}\times\sqrt{0.01^2+0.02^2+0.02^2} = \frac{1}{20} = 0.05$

$\therefore f(1.01, 1.02, -3.98) \simeq f(1,1,-4) + 0.05$

$\simeq 4.05$

a) $f(x) = e^x$

$f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$\vdots \qquad \vdots$

$f^n(x) = e^x \quad f^n(0) = 1$

$\therefore \; C_n = \frac{1}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-0)^n}{n!} = \sum_{n=0} \frac{x^n}{n!}$$

b) $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

Using taylor's theorem

$$f(x) = P_3(x) + R_n(x)$$

error $\simeq R_n(x) = \frac{1}{4!} x^4$

so error $\simeq R(-0.1) = \frac{(0.1)^4}{4!} \approx 4.167 \times 10^{-6}$

$P_3(-0.1) = 1 - 0.1 + \frac{(-0.1)^2}{2} + \frac{(-0.1)^3}{6} \approx 0.904833$

but since $R > 0$ then we have over-estimated

$\therefore \; e^{-0.1} > 0.9$

$e^{0.1} < \frac{1}{0.9} = \frac{10}{9}$

4) $\omega: z^2 - x^2 - y^2 = 0 = f(x,y,z)$

$$\nabla f = (-2x, -2y, 2z)$$

$$\nabla f|_{P_0} = (-6, -8, 10)$$

$$-6(x-3) - 8(y-4) + 10(z-5) = 0$$

$$-6x - 8y + 10z = 0$$

$$3x + 4y - 5z = 0$$

Normal Line: $\begin{cases} x-3 = -6t \\ y-4 = -8t \\ z-5 = 10t \end{cases}$ $t \in \mathbb{R}$

ii) Any normal line has following form vector $\vec{u}(-2x, -2y, 2z)$ as a direction vector.

$$\therefore \vec{u}.\vec{v} = 0$$

$$-2x(1) - 2y(1) + 2z(\sqrt{2}) = 0$$

$$x + y - \sqrt{2}z = 0$$

$$x + y = \sqrt{2}z$$

$(x+y)^2 = 2z^2$ but $z^2 = x^2 + y^2$

$$(x+y)^2 = 2(x^2+y^2)$$

$$x^2 + y^2 + 2xy = 2x^2 + 2y^2$$

$$x^2 - 2xy + y^2 = 0$$

$$(x-y)^2 = 0$$

so $x = y$

$z^2 = x^2 + y^2$

$z^2 = 2x^2$

$z = \sqrt{2}x$ and $z = \sqrt{2}y$

The intersection of 2 planes is a line

iii)

$z = \sqrt{2}y \cap y = x$

let $y = t \quad \dots t \in \mathbb{R}$

$(L): \begin{cases} x = t \\ y = t \\ z = \sqrt{2}t \end{cases}$