MATH 202

- Cylindrical / Rectangular:

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

- Spherical / Rectangular:

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$\rho = \sqrt{r^2+z^2}$
$= \sqrt{x^2+y^2+z^2}$

- Spherical / Cylindrical:

$r = \rho\sin\phi$
$z = \rho\cos\phi$
$\theta = \theta$

$$dV = \begin{cases} dx\,dy\,dz & \text{(rectangular)} \\ dz\,r\,dr\,d\theta & \text{(cylindrical)} \\ \rho^2\sin\phi\,d\rho\,d\phi\,d\theta & \text{(spherical)} \end{cases}$$

$\int_C F\cdot n\,ds$: flux: $\int_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dx\,dy$

$\int_C F\cdot T\,ds$: circulation: $\int_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$

$d\sigma = |r_u \times r_v|\, du\, dv$

$SA = \iint_S d\sigma$

$SA = \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA$

in 2D: $\text{flux} = \int_C F \cdot n\, ds$

in 3D: $\text{flux} = \iint_S F \cdot n\, d\sigma$

$n = \frac{\nabla f}{|\nabla f|}$ or $n = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$

circulation:

by stokes' theorem:

$\int_C F \cdot dr = \iint_S \nabla \times F\, n\, d\sigma$

flux:

by divergence theorem:

$\iint_S F \cdot n\, d\sigma = \iiint \nabla \cdot F\, dV$

$V_{sphere} = \frac{4}{3}\pi r^3$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\int \left(\frac{1}{\cos^2 x}\right) dx = \tan x$$

$$\int \left(\frac{1}{\sin^2 x}\right) dx = -\cot x$$

$$\int \tan x \, dx = -\ln|\cos x| = \ln|\sec x|$$

$$\int \cot x \, dx = \ln|\sin x|$$

$$\int \sec x \, dx = \ln|\sec x + \tan x|$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x|$$

$$\int \ln x \, dx = x \ln x - x$$

$$\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$$

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① Seperable Variables:

$h(y)\,dy = g(x)\,dx$

integrate both sides.

① $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$

② $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

③ $\displaystyle\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + c$

② Linear DE:

$\dfrac{dy}{dx} + P(x)\,y = f(x)$

$\mu = e^{\int P\,dx}$

multiply by μ

$[\mu y]' = f(x)\mu$

integrate

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③ Exact equation:

$M_{(x,y)}\,dx + N_{(x,y)}\,dy = 0$

exact $\Rightarrow$ $f_x = M$, $f_y = N$

$M_y = N_x$ $\Rightarrow$ find function

Not exact $\Rightarrow$ find integrating factor.

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{-M}\, dy$$

or/

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N}\, dx$$

multiply M and N by μ

$\Rightarrow$ find function.

④. Homogeneous Equations.

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

$M(x,y)$ and $N(x,y)$ are hom. functions of same degree.

Use substitution: $y = ux$ or $x = vy$

⑤. Bernoulli's Equation.

$$\frac{dy}{dx} + P(x)\, y = f(x)\, y^n$$

use $u = y^{1-n}$

find $\frac{du}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$

multiply de by this.

$\Rightarrow$ it becomes linear $\Rightarrow$ find integrating factor.

⑥. Ricotti's Equation. substitute: $y = y_1 + u$

given solution

$\Rightarrow$ becomes Bernoulli's

$\Rightarrow$ linear de.

① Reduction of Order: $y'' + P(x)y' + Q(x)y = 0$

→ homogeneous: given y_1

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

→ heterogeneous: change to linear

1. $y = u y_1$
2. find y' and y''
3. replace
4. take $w = u'$ (to make linear of order one)
5. solve using integrating factor.

② Homogeneous linear de with constant coefficients:

$ay'' + by' + cy = 0$

take $y = e^{mx} \Rightarrow y' = me^{mx} \Rightarrow y'' = m^2 e^{mx}$

$\Rightarrow e^{mx}(am^2 + bm + c) = 0$

① two diff real roots: $b^2 - 4ac > 0$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

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② two equal roots: $b^2 - 4ac = 0$

$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

③ complex roots: $b^2 - 4ac < 0$

$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$

③ **Method of undetermined coefficients:**

1. $g(x) = c \Rightarrow y_p = A$
2. $g(x) = ax^2 + c \Rightarrow y_p = Ax^2 + Bx + C$
3. $g(x) = \sin ax \Rightarrow y_p = A\cos ax + B\sin ax$
4. $g(x) = e^{ax} \Rightarrow y_p = Ae^{ax}$
5. $g(x) = x^2 e^{ax} \Rightarrow y_p = (Ax^2 + Bx + C)e^{ax}$.

(if there is duplication, multiply by x)

④ **Variation of parameters:**

$$y_p = u_1 y_1 + u_2 y_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

Trig. Substitution:

1. $\sqrt{a^2 - x^2}$ let $x = a\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$
2. $\sqrt{a^2 + x^2}$ let $x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$
3. $\sqrt{x^2 - a^2}$ let $x = a\sec\theta$ $0 \leq \theta < \pi/2$ & $\pi/2 < \theta \leq \pi$

$$\int \sec x \, dx = \ln|\sec x + \tan x|$$

$$\int \ln x \, dx = x\ln x - x$$

5. Cauchy-Euler Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \text{---}$$

Use $y = x^m$

① $m_1 \neq m_2 \Rightarrow y = c_1 x^{m_1} + c_2 x^{m_2}$

② $m_1 = m_2 \Rightarrow y = c_1 x^{m_1} + c_2 \ln x \, x^{m_1}$

③ $m = \alpha \pm i\beta \Rightarrow y = x^{\alpha} \left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right]$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- **Power Series:** $\sum_{0}^{\infty} C_n (x-a)^n$

① ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

radius of convergence: $|x-a| < R$

- $e^x = \sum_{0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$
- $\sin x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cos x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

- **Multiplication:**

$$C_0 = a_0 b_0$$
$$C_1 = a_1 b_0 + a_0 b_1$$
$$C_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$$
$$\vdots$$

- **Frobenius method:** if $x = x_0$ is a regular singularity

$$y = \sum_{0}^{\infty} C_n (x - x_0)^{n+r}$$

indicial eqn: $r(r-1) + p_0 r + q_0 = 0$

$p = (x - x_0) P$
$q = (x - x_0)^2 Q$ analytic at x_0

① $r_1 \neq r_2$ $r_1 - r_2 \neq N$

$$y_1 = \sum_{0}^{\infty} c_n x^{n+r_1}$$
$$y_2 = \sum_{0}^{\infty} b_n x^{n+r_2}$$

(2) $r_1 \neq r_2$ $\quad r_1 - r_2 = N$

$$y_1 = \sum_0^\infty C_n x^{n+r_1}$$

$$y_2 = C y_1 \ln x + \sum_0^\infty b_n x^{n+r_2}$$

(3) $r_1 = r_2$

$$y_1 = \sum_0^\infty C_n x^{n+r_1}$$

$$y_2 = y_1 \ln x + \sum_0^\infty b_n x^{n+r_1}$$

Laplace:

$$\mathcal{L}\{c\} = \frac{c}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin h\, kt\} = \frac{k}{s^2-k^2}$$

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$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

$$\sinh kt = \frac{1}{2}\left[e^{kt} - e^{-kt}\right]$$

$$\cosh kt = \frac{1}{2}\left[e^{kt} + e^{kt}\right]$$

- $\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - - - - f^{n-1}(0)$

- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

- $\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$

- $\mathcal{L}\{\int_0^t f(\tau)\, d\tau\} = \dfrac{F(s)}{s}$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

. $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

. $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

. $\mathcal{L}\{g(t)\, u(t-a)\} = e^{-as} \mathcal{L}\{g(t+a)\}$

. $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\lim_{s \to \infty} F(s) = 0$$

Eigenvectors:

① **Distinct:** $X = c_1 k_1 e^{\lambda_1 t} + c_2 k_2 e^{\lambda_2 t}$

② **Repeated:**

① multiplicity two $\rightarrow$ two eigenvectors ✓

$\downarrow$

one eigenvector

$$x_1 = k e^{\lambda t}$$

$$x_2 = k t e^{\lambda t} + P e^{\lambda t}$$

$$(A - \lambda I) P = k$$

② multiplicity three:

$\downarrow$

one eigenvector:

$$X = c_1 k e^{\lambda t} + c_2 \left[k t e^{\lambda t} + P e^{\lambda t} \right] + c_3 \left[k \frac{t^2}{2} e^{\lambda t} + P t e^{\lambda t} + Q e^{\lambda t} \right]$$

$$(A - \lambda I) k = 0$$

$$(A - \lambda I) P = k$$

$$(A - \lambda I) Q = P$$