

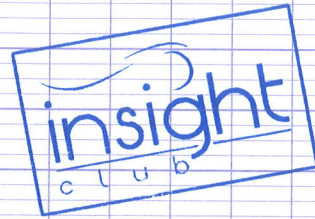
# MATH 202

## • Cylindrical / Rectangular:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$



## • Spherical / Rectangular:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

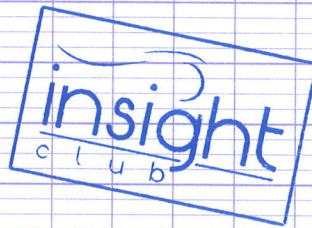
$$\begin{aligned} \rho &= \sqrt{r^2 + z^2} \\ &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

## • Spherical / Cylindrical:

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

$$\theta = \theta$$



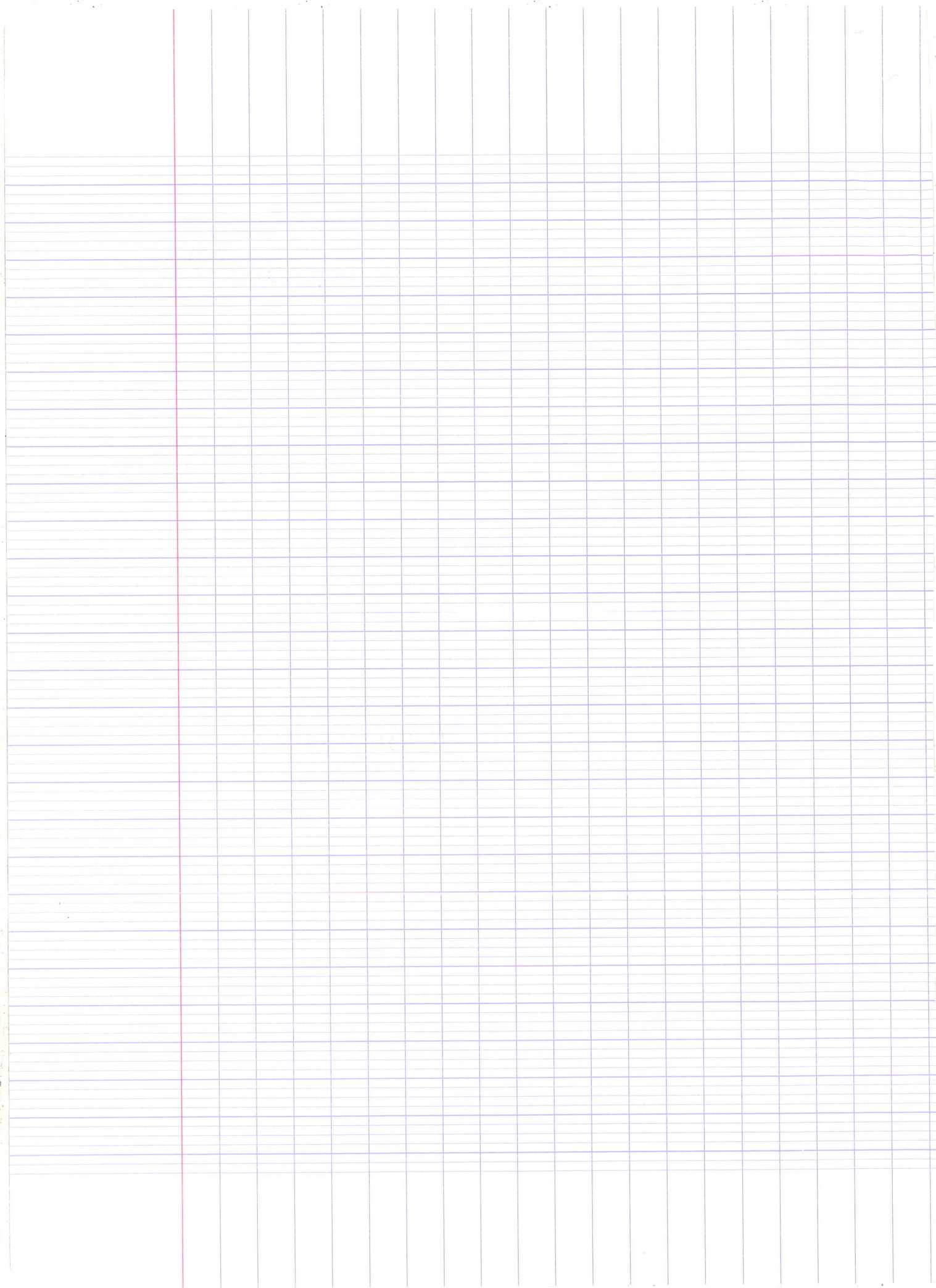
$$dV = \left. \begin{array}{l} dx dy dz \quad (\text{rectangular}) \\ dz r dr d\theta \quad (\text{cylindrical}) \\ \rho^2 \sin \phi d\rho d\phi d\theta \quad (\text{spherical}) \end{array} \right\}$$

$\int_C \mathbf{F} \cdot d\mathbf{s}$  flux:  $\int_C M dy - N dx = \iint_R (M_x + N_y) dx dy$

circulation:  $\int_C M dx + N dy = \iint_R (N_x - M_y) dx dy$

$\int_C \mathbf{F} \cdot T ds$ :







$$d\sigma = |r_u \times r_v| du dv$$



$$SA = \iint_S d\sigma$$

$$SA = \iint_R \frac{|\nabla F|}{|\nabla F \cdot P|} dA$$

in 2D: Flux =  $\int_C F \cdot n ds$

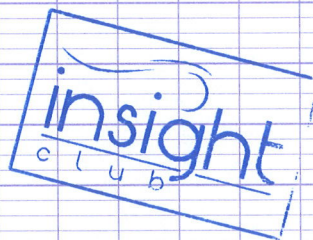
in 3D: Flux =  $\iint_S F \cdot n d\sigma$

$$n = \frac{\nabla f}{|\nabla f|} \quad \text{or} \quad n = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$$

circulation:

by Stokes' theorem:

$$\int_C F \cdot dr = \iint_S \text{curl } F \cdot n d\sigma$$



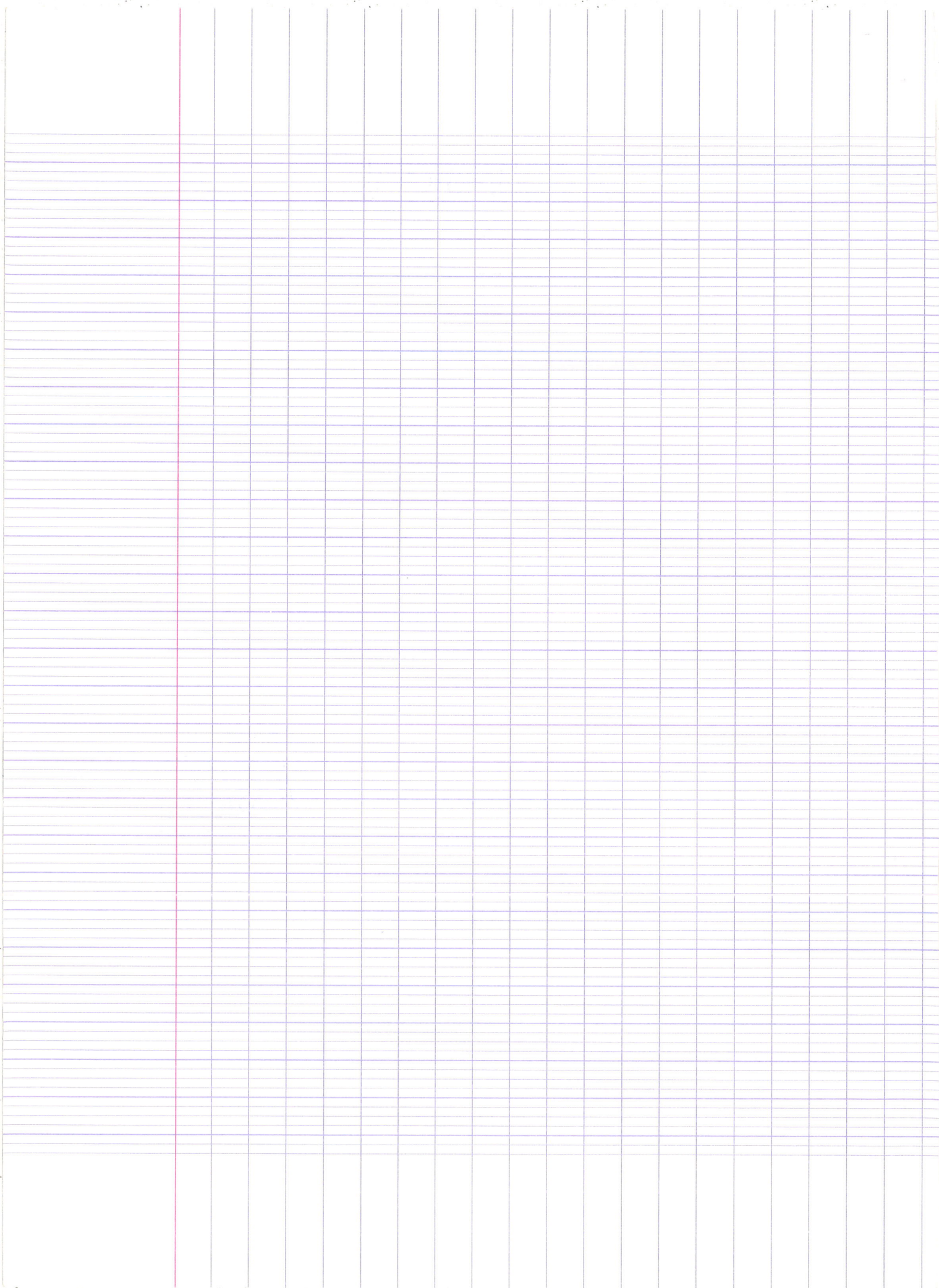
flux:

by divergence theorem:

$$\iint_S F \cdot n d\sigma = \iiint_V \text{div } F dV$$

$$V_{\text{sphere}} = \frac{4}{3} \pi r^3$$







$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\int \left( \frac{1}{\cos^2 x} \right) dx = \tan x$$

$$\int \left( \frac{1}{\sin^2 x} \right) dx = -\cot x$$

$$\int \tan x \, dx = -\ln |\cos x| = \ln |\sec x|$$

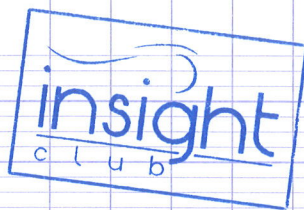
$$\int \cot x \, dx = \ln |\sin x|$$

$$\int \sec x \, dx = \ln |\sec x + \tan x|$$

$$\int \csc x \, dx = -\ln |\csc x + \cot x|$$

$$\int \ln x \, dx = x \ln x - x$$

$$\left( \tan^{-1}(x) \right)' = \frac{1}{1+x^2}$$

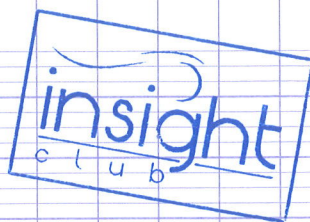








### ①. Separable Variables:



$$h(y) dy = g(x) dx$$

integrate both sides.

$$① \int \frac{dx}{1-x^2} = \sin^{-1} x + C$$

$$② \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$③ \int \frac{dx}{|x| \sqrt{x^2-1}} = \sec^{-1} x + C$$

### ②. Linear DE:

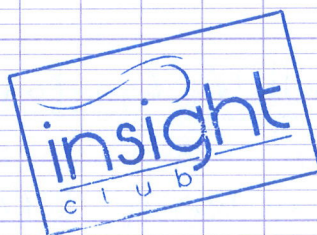
$$\frac{dy}{dx} + P(x)y = f(x)$$

$$\mu = e^{\int P dx}$$

multiply by  $\mu$

$$[\mu y]' = f(x)\mu$$

integrate



### ③. Exact equation:

$$M(x,y) dx + N(x,y) dy = 0$$

$$\text{exact} \Rightarrow f_x = M, f_y = N$$

$$M_y = N_x \Rightarrow \text{find function}$$



Not exact  $\Rightarrow$  find integrating factor.

$$\frac{dm}{m} = \frac{My - Nx}{-N} dy$$

$$\text{or } \frac{dm}{m} = \frac{My - Nx}{N} dx$$

multiply M and N by  $m$   
 $\Rightarrow$  find function.



#### (4) Homogeneous Equations.

$$M(x,y) dx + N(x,y) dy = 0$$

$M(x,y)$  and  $N(x,y)$  are hom. functions of same degree.

Use substitution:  $y = ux$  or  $x = vy$

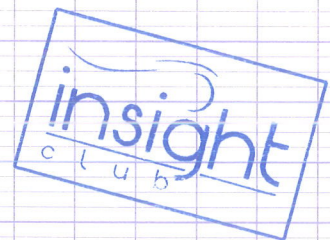
#### (5) Bernoulli's Equation.

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

use  $u = y^{1-n}$   
find  $\frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx}$

multiply de by this.

$\Rightarrow$  it becomes linear  $\Rightarrow$  find integrating factor.



#### (6) Riccati's Equation.

substitute:  $y = (y_1) + u$   
given solution.

$\Rightarrow$  becomes Bernoulli's  
 $\Rightarrow$  linear de.



① Reduction of Order:  $y'' + P(x)y' + Q(x)y = 0$

→ homogeneous: given  $y_1$   
 $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

→ heterogeneous: change to linear

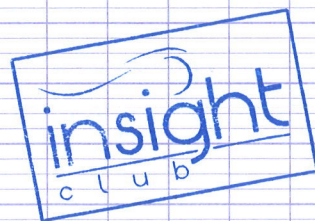
1-  $y = uy_1$

2- find  $y'$  and  $y''$

3- replace

4- take  $w = u'$  (to make linear of order one)

5- solve using integrating factor



② Homogeneous linear de with constant coefficients:

$$ay'' + by' + cy = 0$$

$$\text{take } y = e^{mx} \Rightarrow y' = me^{mx} \Rightarrow y'' = m^2e^{mx}$$

$$\Rightarrow e^{mx}(am^2 + bm + c) = 0$$

① two diff real roots:  $b^2 - 4ac > 0$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

② two equal roots:  $b^2 - 4ac = 0$

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

③ complex roots:  $b^2 - 4ac < 0$

$$y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$$





③ Method of undetermined coefficients:

1.  $g(x) = c \Rightarrow y_p = A$

2.  $g(x) = ax^2 + c \Rightarrow y_p = Ax^2 + Bx + C$

3.  $g(x) = \sin ax \Rightarrow y_p = A \cos ax + B \sin ax$

4.  $g(x) = e^{ax} \Rightarrow y_p = Ae^{ax}$

5.  $g(x) = x^2 e^{ax} \Rightarrow y_p = (Ax^2 + Bx + C)e^{ax}$

(if there is duplication, multiply by  $x$ )

④ Variation of parameters:



$$y_p = u_1 y_1 + u_2 y_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$u_1' = \frac{W_1}{W}$$

$$u_2' = \frac{W_2}{W}$$

Trig. Substitution:



1.  $\sqrt{a^2 - x^2}$  let  $x = a \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

2.  $\sqrt{a^2 + x^2}$  let  $x = a \tan \theta$ ,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

3.  $\sqrt{x^2 - a^2}$  let  $x = a \sec \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$  &  $\frac{\pi}{2} < \theta \leq \pi$

$$\int \sec x \, dx = \ln |\sec x + \tan x|$$

$$\int \ln x \, dx = x \ln x - x$$



⑤ Cauchy-Euler Equation.

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots$$



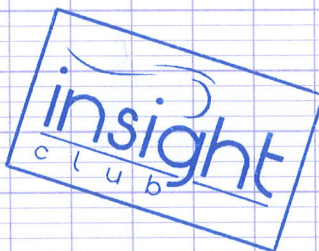
Use  $y = x^m$

①  $m_1 \neq m_2 \Rightarrow y = c_1 x^{m_1} + c_2 x^{m_2}$

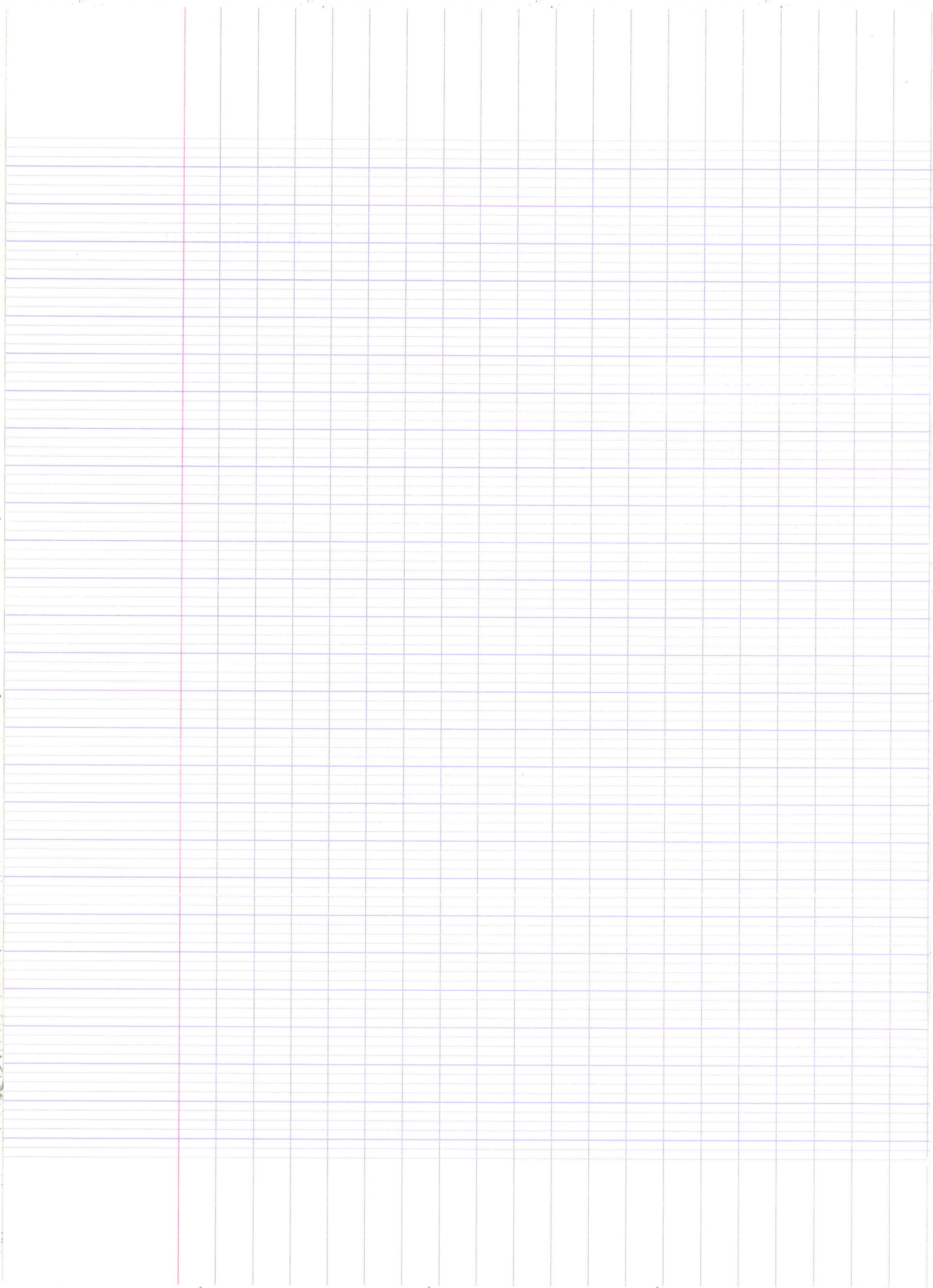
②  $m_1 = m_2 \Rightarrow y = c_1 x^{m_1} + c_2 \ln x x^{m_1}$

③  $m = \alpha \pm i\beta \Rightarrow y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



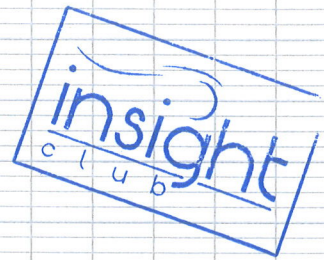






• Power Series:  $\sum_0^{\infty} C_n (x-a)^n$

① ratio test:  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$



radius of convergence:  $|x-a| < R$

•  $e^x = \sum_0^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$

•  $\sin x = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

•  $\cos x = \sum_0^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

- Multiplication:  
 $C_0 = a_0 b_0$   
 $C_1 = a_1 b_0 + a_0 b_1$   
 $C_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$   
 $\vdots$

• Frobenius method: if  $x=x_0$  is a regular singularity  
 $y = \sum_0^{\infty} C_n (x-x_0)^{n+r}$

indicial eqn:  $r(r-1) + p_0 r + q_0 = 0$

$$p = (x-x_0)P$$
$$q = (x-x_0)^2 Q$$

analytic at  $x_0$



①  $r_1 \neq r_2$   $r_1 - r_2 \neq N$   
 $y_1 = \sum_0^{\infty} C_n x^{n+r_1}$   
 $y_2 = \sum_0^{\infty} b_n x^{n+r_2}$



$$\textcircled{2} \quad r_1 \neq r_2 \quad r_1 - r_2 = N$$
$$y_1 = \sum_0^{\infty} C_n x^{n+r_1}$$

$$y_2 = C y_1 \ln x + \sum_0^{\infty} b_n x^{n+r_2}$$

$$\textcircled{3} \quad r_1 = r_2$$
$$y_1 = \sum_0^{\infty} C_n x^{n+r_1}$$

$$y_2 = y_1 \ln x + \sum_0^{\infty} b_n x^{n+r_1}$$





Laplace:

$$\mathcal{L}\{c\} = \frac{c}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin hkt\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

$$\sinh kt = \frac{1}{2} [e^{kt} - e^{-kt}]$$

$$\cosh kt = \frac{1}{2} [e^{kt} + e^{-kt}]$$

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{n-1}(0)$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$$

$$\mathcal{L}\left\{\int_0^t f(z) dz\right\} = \frac{F(s)}{s}$$





$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$\cdot \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

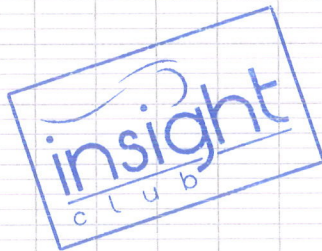
$$\cdot \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\cdot \mathcal{L}\{g(t)u(t-a)\} = e^{-as} \mathcal{L}\{g(t+a)\}$$

$$\cdot \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$$

$$\lim_{s \rightarrow \infty} F(s) = 0$$





## Eigenvectors:



① Distinct:  $X = c_1 k_1 e^{\lambda_1 t} + c_2 k_2 e^{\lambda_2 t}$

## ② Repeated:

① multiply two  $\rightarrow$  two eigenvectors  $\checkmark$



one eigenvector

$$x_1 = k e^{\lambda t}$$

$$x_2 = k t e^{\lambda t} + p e^{\lambda t}$$

$$(A - \lambda I)P = K$$

② multiply three:



one eigen vector:

$$X = c_1 k e^{\lambda t} + c_2 [k t e^{\lambda t} + p e^{\lambda t}]$$

$$+ c_3 [k \frac{t^2}{2} e^{\lambda t} + p t e^{\lambda t} + q e^{\lambda t}]$$

$$(A - \lambda I)K = 0$$

$$(A - \lambda I)P = K$$

$$(A - \lambda I)Q = P$$





