MATH 201: Calculus and Analytic Geometry III

Fall 2016-2017, Exam 1, Duration: 60 min.

Problem	1	2	3	4	5	Total
Points	20	28	22	20	10	100
Scores						

Name: Correction AUB ID: ____________________

Please circle your section:

Section 1
MWF 3, Nahlus
Recitation F. 11

Section 2
MWF 3, Nahlus
Recitation F. 10

Section 3
MWF 3, Nahlus
Recitation F. 8

Section 4
MWF 3, Nahlus
Recitation F. 9

Section 5
MWF 10, Shayya
Recitation T. 11

Section 6
MWF 10, Shayya
Recitation T. 12:30

Section 7
MWF 10, Shayya
Recitation T. 2

Section 8
MWF 10, Shayya
Recitation T. 5

Section 9
MWF 11, Yamani
Recitation F. 2

Section 10
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Recitation F 3

Section 11
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Recitation F. 4

Section 12
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Section 13
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Section 14
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Section 15
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Section 16
MWF 2, Nahlus
Recitation M. 8

Section 17
MWF 9, Makdisi
Recitation Th. 9:30

Section 18
MWF 9, Makdisi
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Section 19
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Section 20
MWF 9, Makdisi
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Section 21
MWF 1, Karam
Recitation F. 10

Section 22
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Section 23
MWF 1, Karam
Recitation F. 12

Section 24
MWF 1, Karam
Recitation F. 8

Section 25
MWF 10, AbiKhuzam
Recitation F. 4

Section 26
MWF 10, AbiKhuzam
Recitation F. 2

Section 27
MWF 10, AbiKhuzam
Recitation F 3

Section 28
MWF 10, AbiKhuzan
Recitation F. 1

Section 29
MWF 11, Aoun
Recitation Th. 3:30

Section 30
MWF 11, Aoun
Recitation Th. 2

Section 31
MWF 11, Aoun
Recitation Th. 5

Section 32
MWF 11, Aoun
Recitation Th. 12:3(

INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.
(b) Closed book. No notes. No calculators. No cellphones.
(c) UNLESS CLEARLY SPECIFIED OTHERWISE, THE BACKSIDE OF THE PAGES WILL NOT BE GRADED,

Problem 1

(5 pts each)Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = \left(\frac{5n+1}{5n-1}\right)^n$

$$a_n = \left(\frac{\cancel{5n}(1+\frac{1}{5n})}{\cancel{5n}(1-\frac{1}{5n})}\right)^n = \frac{\left(1+\frac{1/5}{n}\right)^n}{\left(1+\frac{-1/5}{n}\right)^n}$$

$$\longrightarrow \frac{e^{1/5}}{e^{-\frac{1}{5}}} = \boxed{e^{2/5}}$$ (So $\{a_n\}$ is convergent)

This uses $\left(1+\frac{a}{n}\right)^n \longrightarrow e^a$ (basic limit).

(b) $b_n = n\left(7^{1/n} - 1\right)$

$$b_n = \frac{7^{\frac{1}{n}} - 1}{\frac{1}{n}}.\quad \text{As } n \to \infty,\ \frac{1}{n} \to 0,$$

so let us study whether $\lim\limits_{x \to 0} \dfrac{7^x - 1}{x}$ exists.

Now as $x \to 0$, $7^x - 1 \to 7^0 - 1 = 0$ so this is a limit of the form "$\frac{0}{0}$". We can use L'Hôpital's rule:

Remember $7^x = e^{x \ln 7}$ so $(7^x)' = \underbrace{(e^{x\ln 7})' \cdot \ln 7}_{\text{chain rule}} = 7^x \ln 7.$

Here we study $\lim\limits_{x \to 0} \dfrac{(7^x - 1)'}{(x)'} = \lim\limits_{x \to 0} \dfrac{7^x \ln 7}{1} = 7^0 \ln 7 = \boxed{\ln 7}$

(7^x is continuous at $x = 0$.)

Since this limit exists, we deduce $\boxed{b_n \longrightarrow \ln 7}$ (also convergent).

<u>Note</u> using Taylor series (not on this exam)

$$7^{\frac{1}{n}} = e^{\frac{1}{n}\ln 7} = 1 + \left(\tfrac{1}{n}\right)\ln 7 + \frac{\left(\frac{1}{n^2}\right)(\ln 7)^2}{2!} + \frac{\left(\frac{1}{n^3}\right)(\ln 7)^3}{3!} + \cdots$$

$$= 1 + \left(\tfrac{1}{n}\right)\ln 7 + \left(\text{quantity bounded by } \tfrac{C}{n^2}\right)$$

Then $7^{\frac{1}{n}} - 1 = \left(\frac{1}{n}\ln 7\right) + \left(\text{quantity bounded by } \frac{C}{n^2}\right)$

$$n\left(7^{\frac{1}{n}} - 1\right) = (\ln 7) + \underbrace{\left(\text{quantity bounded by } \tfrac{C}{n}\right)}_{\to 0 \text{ as } n \to \infty} \longrightarrow \ln 7$$

as $n \to \infty$.

(c) $c_n = \dfrac{n^{1/n} cos\left(5+n^3\right)}{\sqrt{n}}$

Here: $c_n = \left(n^{\frac{1}{n}}\right) \cdot \dfrac{\cos(5+n^3)}{\sqrt{n}} = n^{\frac{1}{n}} \cdot e_n$

where $n^{\frac{1}{n}} \longrightarrow 1$ (basic limit)

and $e_n = \dfrac{\cos(5+n^3)}{\sqrt{n}}$; since $\forall n$, $-1 \leq \cos(5+n^3) \leq 1$,

we have $\forall n$, $\dfrac{-1}{\sqrt{n}} \leq e_n \leq \dfrac{1}{\sqrt{n}}$.

But $\dfrac{-1}{\sqrt{n}}$ and $\dfrac{1}{\sqrt{n}}$ both converge to the <u>same</u> limit 0,

so by the sandwich theorem, $e_n \longrightarrow 0$.

Thus $\lim c_n = \left(\lim n^{\frac{1}{n}}\right) \cdot \left(\lim e_n\right) = 1 \cdot 0 = \boxed{0}$ ($\{c_n\}$ is convergent.)

(d) $d_n = \left(n + (-1)^n n\right)^{1/n}$

For n even, $n = 2k$, we have

$$d_{2k} = \left(2k + (+1)\cdot 2k\right)^{\frac{1}{2k}} = (4k)^{\frac{1}{2k}}$$

$$= \underbrace{2^{\frac{1}{2k}}}_{\substack{\text{this} \to 1 \\ \text{as } k \to \infty \\ (\text{like } a^{1/n} \to 1)}} \cdot \underbrace{(2k)^{\frac{1}{2k}}}_{\substack{\text{this} \to 1 \\ \text{as } k \to \infty \\ (\text{like } n^{1/n} \to 1)}}$$

so $\boxed{k \to \infty \Rightarrow d_{2k} \to 1.}$

On the other hand, for n odd, $n = 2k+1$, we have

$$d_{2k+1} = \left((2k+1) + (-1)(2k+1)\right)^{\frac{1}{2k+1}}$$

$$= (0)^{\frac{1}{2k+1}} = 0$$

so $\boxed{k \to \infty \Rightarrow d_{2k} \to 0.}$

The above shows that two different subsequences of $\{d_n\}$ have different limits (1 and 0). This shows that $\{d_n\}$ diverges. However, we did not see the theorem on subsequences in class. So here is another way to see it:

Suppose that a limit existed, so $d_n \to L$ for some L. Choosing $\varepsilon = 0.1$ (for example: any $\varepsilon < \frac{1}{2}$ works here), we have that for all large n, $L - 0.1 < d_n < L + 0.1$.

short interval supposedly containing all the d_n's for large n

In particular, for n_1 = large and even
n_2 = large and odd

we have $d_{n_1}, d_{n_2} \in (L-0.1, L+0.1)$ so d_{n_1} and d_{n_2} are at most 0.2 apart, ($|d_{n_1} - d_{n_2}| < 0.2$). But $d_{n_1} \approx 1$ and $d_{n_2} \approx 0$ so this is impossible. This contradiction shows that $d_n \not\to L$.

Problem 2

(7 pts each) Which of the following series converge, and which diverge?

Find the sum of the series when possible.

(a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3^{n-1}}+\frac{2^n}{3^{n+2}}\right)$

Let $S_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n-1}} = \sum_{n=0}^{\infty} 3\left(\frac{-1}{3}\right)^n = 3 - 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3^2}\right) - \cdots$

It's a geometric series with $r = -\frac{1}{3}$. $|r|<1$ so S_1 converges and its value is $\boxed{S_1 = 3\left(1 + \left(-\frac{1}{3}\right) + \left(\frac{-1}{3}\right)^2 + \cdots\right) = \frac{3}{1-\left(\frac{-1}{3}\right)} = \frac{9}{4}}$.

Let $S_2 = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{1}{9}\left(\frac{2}{3}\right)^n$. This is also a geometric series, this time with $r = \frac{2}{3} < 1$, so S_2 converges and $\boxed{S_2 = \frac{1}{9}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}}$.

Our series is the sum of two convergent series, so our original series converges, and its sum is

$$S_1 + S_2 = \frac{9}{4} + \frac{1}{3} = \frac{31}{12}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4+\sin\left(n^{10}\right)}{n\sqrt{n}} = \sum_{n=1}^{\infty} a_n$

We know $-1 \le \sin(n^{10}) \le 1$ for all n,

so (adding 4 to the inequalities),

$3 \le 4+\sin(n^{10}) \le 5$ for all n.

Thus

$$\underbrace{0 < \frac{3}{n\sqrt{n}}} \le a_n = \frac{4+\sin(n^{10})}{n\sqrt{n}} \le \frac{5}{n\sqrt{n}} \quad \text{for all } n.$$

this side shows that $a_n \ge 0$ for all n, so $\sum a_n$ is a series of nonnegative terms. We can therefore use a comparison theorem. In this example we use the direct comparison theorem with the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n}}$, because $a_n \le \frac{5}{n\sqrt{n}}$ for all n.

Since the "larger" series $\sum \frac{5}{n\sqrt{n}}$ converges (it's essentially a p series with $p=1.5>1$), the "smaller" series $\sum a_n$ $\boxed{\text{converges}}$ also.

(c) $\sum_{n=1}^{\infty} \frac{(n-4)\sqrt{n}}{(n+1)^2} = \sum a_n$

here write $a_n = \frac{n\sqrt{n} - 4\sqrt{n}}{n^2+2n+1} = \frac{n\sqrt{n}}{n^2} \cdot \frac{\left(1 - \frac{4}{n}\right)}{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}$,

and compare $\sum a_n$ to $\sum b_n$, where $b_n = \frac{n\sqrt{n}}{n^2} = \frac{1}{\sqrt{n}}$.

We have $a_n \geq 0$ for $n \geq 4$ and $b_n \geq 0$ for all n, so we can use a comparison theorem. Here $\frac{a_n}{b_n} = \frac{1-\frac{4}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} \to 1 \neq 0, \infty$

So the limit comparison theorem says $\sum a_n$ & $\sum b_n$ either both converge or both diverge.

However $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges (p-series with $p = 0.5 < 1$)

So $\boxed{\sum a_n \text{ diverges}}$

(d) $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^n = \sum a_n$

here $a_n = \left(1-\frac{2}{n}\right)^n \longrightarrow e^{-2}$ by one of our basic limits.

Note $\boxed{e^{-2} \neq 0}$ so the a_n does NOT converge to zero

so by the nth term test, $\sum a_n$ does NOT converge.

Note 1: the sequence $\{a_n\} = \left\{\left(1-\frac{2}{n}\right)^n\right\}$ does converges, namely to e^{-2}

but the series $\sum_1^{\infty} a_n = \sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^n$ diverges.

Note 2 For $\sum a_n$ to converge, it is necessary for $a_n \to 0$ (that's the nth term test), but it is not sufficient: The series $\sum \frac{1}{n}$ diverges, even though $\frac{1}{n} \to 0$.

Problem 3
(a)(14 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)2^n}(x-1)^n$$

(Remember to check the endpoints.)

Write this as $\sum a_n$ with $a_n = \frac{(x-1)^n}{(n^2+1)\,2^n}$

& apply the ratio test to $\sum |a_n|$: [(the root test works just as well: for the root test, use

$$\boxed{1 = \sqrt[n]{1} \le \sqrt[n]{n^2+1} \le \sqrt[n]{2n^2} = \sqrt[n]{2}\,(\sqrt[n]{n})^2}$$

So calculate

$$\rho = \lim \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim \left| \frac{(x-1)^{n+1}}{((n+1)^2+1)\,2^{n+1}} \cdot \frac{(n^2+1)\,2^n}{(x-1)^n} \right|$$

$$= \lim \frac{n^2+1}{n^2+2n+2} \cdot \frac{|x-1|}{2} = \lim \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \cdot \frac{|x-1|}{2} = \frac{|x-1|}{2}.$$

The open interval of convergence is when $\rho < 1$, ie, $\frac{|x-1|}{2} < 1$,

ie, $|x-1| < 2$, ie $\boxed{2-1 < x < 2+1}$, ($2-1 = 1$, $2+1 = 3$)

Endpoints: when $x = 2+1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)} \frac{\cancel{2^n}}{\cancel{2^n}}$

$= \sum \frac{1}{n^2+1}$, which converges by DCT to the p-series $\sum \frac{1}{n^2}$.

When $x = 2-1$, we get $\sum \frac{1}{(n^2+1)} \cdot \frac{(-2)^n}{2^n} = \sum \frac{(-1)^n}{n^2+1}$,

which converges absolutely b/c $\sum \frac{1}{n^2+1}$ converges (or use alternating series),

So the interval of convergence includes BOTH endpoints.

$\boxed{I = [1,3]}$.

(b) (8 pts) For $x = 0.6$, use the alternating series estimation theorem (ASET) to approximate the sum of the series in part **(a)** with an error of magnitude no greater than 10^{-2}. (Make sure to justify why the conditions for ASET are satisfied.)

For $x=0.6$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-0.4)^n}{(n^2+1)(2^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n (0.2)^n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n u_n,$$

where $u_n = \frac{(0.2)^n}{n^2+1}$ $\underline{\geq 0 \text{ for all } n}$, so the series $\sum_{n=1}^{\infty}(-1)^n u_n$

is indeed alternating. (1)

We must check if u_n is decreasing: as n increases,

$(0.2)^n$ decreases (powers of 0.2, with $0<0.2<1$), and

(n^2+1) increases (because n^2 increases with n, or use derivatives)

So $\underline{u_n = \frac{(0.2)^n}{n^2+1} \text{ decreases as } n \text{ increases}}$, (2) (numerator ↓ and denominator ↑.)

Third, $u_n \to 0$, for example by noting $n^2+1 \geq 1$ and sandwiching

$$0 < u_n = \frac{(0.2)^n}{n^2+1} < (0.2)^n \quad \text{\& noting } (0.2)^n \to 0.$$

(you can also sandwich $0 < u_n < \frac{1}{n^2+1}$) So $\underline{u_n \to 0}$. (3)

This means ASET can be applied. Let $S_N = \sum_{n=1}^{N} (-1)^n u_n$, and

$L = \sum_{n=1}^{\infty} (-1)^n u_n$. Then $\boxed{|S_N - L| < u_{N+1} = \frac{(0.2)^{N+1}}{(N+1)^2+1}}$.

We want an N where $\frac{(0.2)^{N+1}}{(N+1)^2+1} < 10^{-2} = 0.01$,

Try $\boxed{N=1}$, so $u_2 = \frac{(0.2)^2}{(1+1)^2+1} = \frac{0.04}{5} < \frac{0.05}{5} = 0.01$. This works. ($u_2 = 0.008$.)

so $\boxed{L \text{ is approximated by } S_1 = -\frac{(0.2)}{1^2+1} = -0.1 \text{ with an |error|} < 0.01}$

(note L is approximated by $S_2 = \frac{-(0.2)}{1^2+1} + \frac{(0.2)^2}{2^2+1} = -0.092$ with |error| $< u_3 = \frac{(0.2)^3}{3^2+1} = 8 \times 10^{-4}$.)

Problem 4

(10 pts each) For each of the following series, find the n^{th} partial sum s_n. Then decide if the series converges or diverges. Find the sum of the series when possible.

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)$

This is a telescoping series: the Nth partial sum S_N is

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right) = \left(\frac{1}{1^3}-\cancel{\frac{1}{2^3}}\right)+\left(\cancel{\frac{1}{2^3}}-\cancel{\frac{1}{3^3}}\right)+\left(\cancel{\frac{1}{3^3}}-\cancel{\frac{1}{4^3}}\right)+\cancel{\cdots}+\left(\cancel{\frac{1}{N^3}}-\frac{1}{(N+1)^3}\right)$$
$$= \frac{1}{1^3}-\frac{1}{(N+1)^3}.$$

So $\boxed{S_N = 1-\frac{1}{(N+1)^3}}$, $\boxed{\lim_{N\to\infty} S_N = 1-0 = 1}$,

so $\boxed{\text{the series converges, and its value is } \lim S_N = 1.}$

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$

This series is also telescoping, but in a hidden way.

Write $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$

so $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{(n+1)-(n)} = \sqrt{n+1} - \sqrt{n}$.

Our series is therefore $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$

and $S_N = \sum_{n=1}^{N}(\sqrt{n+1}-\sqrt{n}) = (\cancel{\sqrt{2}}-\sqrt{1}) + (\cancel{\sqrt{3}}-\cancel{\sqrt{2}}) + (\cancel{\sqrt{4}}-\cancel{\sqrt{3}}) + \cdots + \cdots + (\sqrt{N+1}-\cancel{\sqrt{N}})$

$= \sqrt{N+1} - \sqrt{1}$

so $S_N = \sqrt{N+1} - 1$

and S_N <u>diverges</u> to $+\infty$ when $N \to \infty$

so <u>$\sum a_n$ diverges</u>.

(note one can compare $\sum a_n$ to the divergent p-series $\sum \frac{1}{\sqrt{n}}$ to see $\sum a_n$ diverges, but the problem specifically asked for the value of S_N.)

Problem 5

(10 pts) Given a series $\sum_{n=1}^{\infty} a_n$ of positive terms, and $\{s_n\}$ its sequence of partial sums. No direct information about the n^{th} term a_n or partial sum s_n is given. But instead we are given that

$$\lim_{n\to\infty} a_n^{\ 2} . s_n = 5.$$

Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

(Hint: start by assuming the series converges, and see what you may conclude.)

This is a proof by contradiction.

Suppose $\sum_{n=1}^{\infty} a_n$ converged, with $L = \sum_{n=1}^{\infty} a_n$.

This means that $\boxed{s_n \to L}$ (the sequence of partial sums converges to L).

But we also have that $\boxed{a_n \to 0}$ because of the nth term test applied to the supposedly convergent series $\sum a_n$.

But then $a_n^2 \cdot s_n \to 0^2 \cdot L = 0 \neq 5$, contradicting the given.

So our supposition was false,

and $\boxed{\sum a_n \text{ must diverge}}$.