Math 201–Exam 2 (Fall 01)

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)
(a) (10 pts) Reverse the order of integration of

$$\int_0^2 \int_0^{x^2/2} 2x(1+x^2+y)\,dy\,dx$$

and evaluate the resulting integral.
(b) (12 pts) Change

$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$$

into polar coordinates and evaluate the resulting integral.

Problem 2 (answer on page 2 of the booklet.)
(18 pts) Find the tangent plane and normal line of the surface

$$z = \ln(2x+3y+6z)$$

at the point $(-1,-1,1)$.

Problem 3 (answer on page 3 of the booklet.)
Suppose $f(x,y,z)$ is a differentiable function. Let

$$x = r^2+s, \qquad y = \frac{r}{s}+1 \qquad z = 2r + \ln s, \qquad \text{and} \qquad w = f(x,y,z).$$

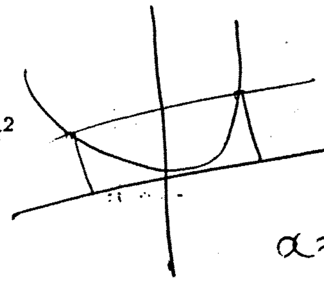
~~(a) (12 pts) Find $\partial w/\partial r$ and $\partial w/\partial s$ in terms of f_x, f_y, and f_z.~~ $dw/dr = 2r f_x + \frac{1}{s} f_y + 2 f_z$
(b) (6 pts) Which of the following choices of ∇f:

dw/ds

$$\nabla f(5,3,4) = \mathbf{i}+2\mathbf{j}+\mathbf{k}, \qquad \nabla f(5,3,4) = \frac{-1}{3}\mathbf{i}+\mathbf{j}-2\mathbf{k}, \qquad \text{or} \qquad \nabla f(2,1,0) = \mathbf{i}+\mathbf{j}-6\mathbf{k}$$

guarantees that $\partial w/\partial r = \partial w/\partial s$ at the point $(r,s) = (2,1)$? Give reasons for your answer.

Problem 4 (answer on pages 4 and 5 of the booklet.)
(20 pts) Find the maximum and minimum values of



$$f(x,y,z) = x^2y^2z^2$$

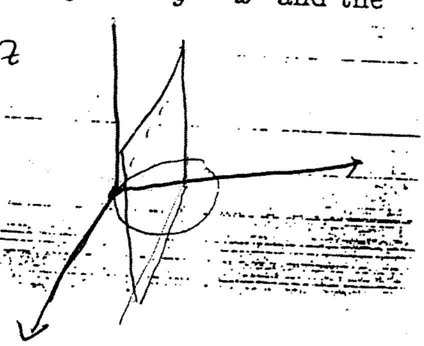
subject to the constraint $x^2+y^2+z^2 = 1$.

$x = \pm\sqrt{y}$

Problem 5 (answer on page 6 of the booklet.)
Let D be the region bounded below by the plane $z=0$, laterally by the cylinder $y = x^2$ and the plane $y=1$, and above by the plane $z = 1+y$.
(a) (12 pts) Find the volume of D.
(b) (10 pts) Set up the limits of integration for evaluating $y = 1-z$

$$\iiint_D f(x,y,z)\,dV$$

as an iterated triple integral in the order $dydzdx$.



$\int_0 \int_{x^2}^{1} \int_0^{1+y}$ $\{z \{y \{x$

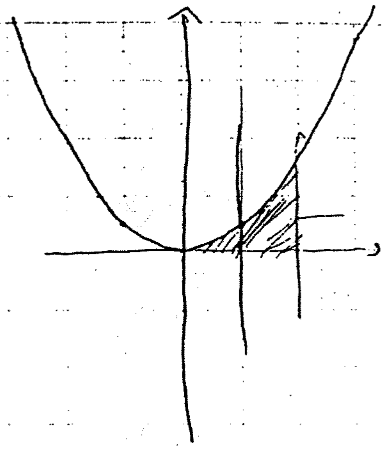
Exam 2 Fall 01

Problem 1

$$\int_0^2 \int_0^{x^2/2} 2x(1+x^2+y)\,dy\,dx$$

$y = \frac{x^2}{2}$

$x = \sqrt{2y}$



$$\int_0^2 \int_{\sqrt{2y}}^2 2x(1+x^2+y)\,dx\,dy$$

$$\int_{\sqrt{2y}}^2 2x + 2x^3 + 2xy\,dx = x^2 + \frac{2x^4}{2} + x^2 y \Big]_{\sqrt{2y}}^2 = 4 + 8 + 4y - 2y - 2y^2 - 2y^2$$
$$= 12 + 2y - 4y^2$$

$$\int_0^2 (12 + 2y - 4y^2)\,dy = 12y + y^2 - \frac{4y^3}{3}\Big]_0^2 = 72 + 4 - 16 = \frac{[illegible]}{3}$$

(b) $$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy = \int_0^{\pi/2} \int_0^1 e^{r^2} r\,dr\,d\theta$$

$$\int_0^1 e^{r^2} r\,dr = \frac{e^{r^2}}{2}\Big]_0^1 = \frac{e}{2} - \frac{1}{2}$$

$$\Rightarrow I = \int_0^{\pi/2} \left(\frac{e}{2} - \frac{1}{2}\right) d\theta = \frac{e\pi}{4} - \frac{\pi}{4}$$

Problem 2

$z = \ln(2x + 3y + 6z)$

Let $f(x,y,z) = \ln(2x+3y+6z) - z$

$\Rightarrow z = \ln(2x+3y+6z)$ is level surface $f(x,y,z) = 0$

Fall 01

(a) $x = r^2 + s \qquad y = \frac{r}{s} + 1 \qquad z = 2r + \ln s \qquad w = f(x,y,z)$

$$\frac{\partial w}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} + f_z \frac{\partial z}{\partial r}$$

$$= f_x(2r) + f_y\left(\frac{1}{s}\right) + f_z(2)$$

$$= 2r(f_x) + \frac{f_y}{s} + 2f_z$$

$$\frac{\partial w}{\partial s} = f_x - \frac{r f_y}{s^2} + \frac{f_z}{s}$$

(b) $(r,s) = (2,1) \Rightarrow \frac{\partial w}{\partial r} = 4f_x + f_y + 2f_z$

$\quad \& \ \frac{\partial w}{\partial s} = f_x - 2f_y + f_z$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial r}$

$\Rightarrow 3f_x + 3f_y + f_z = 0$

for $r = 2$ & $s = 1 \quad x = 5 \quad y = 3 \quad z = 4$

$\Rightarrow \nabla f(2,1,0) = i + j - 6k$. is omitted from choices.

$\nabla f(5,3,4) = i + 2j + k \Rightarrow f_x = 1 \quad f_y = 2 \quad f_z = 1$

but $3f_x + 3f_y + f_z = 0$

$3 + 6 + 1 \neq 0 \Rightarrow \nabla f(5,3,4) = \cancel{\ } i + 2j + k$ does n satisfy the relation.

$\nabla f(5,3,4) = -\frac{1}{3} i + j - 2k \Rightarrow f_x = -\frac{1}{3} \quad f_y = 1 \quad f_z = -2$

$\Rightarrow 3f_x + 3f_y + f_z = -1 + 3 - 2 = 0 \Rightarrow$ satisfies relation

$\Rightarrow \nabla f(5,3,4)$ is the choice which guarantees the given con.

at (2,1) $\dfrac{\partial w}{\partial r} = 4f_x + f_y + 2f_z$

$\dfrac{\partial w}{\partial s} = f_x - 2f_y + f_z$

1st case $\nabla f(5,3,4) = i + 2j + k \Rightarrow \dfrac{\partial w}{\partial r} = 4 + 2 + 2 = 8$

$\dfrac{\partial w}{\partial s} = 4 - 4 + 1 = 2 \Rightarrow$ not equal.

at $\nabla f(5,3,4) = -\frac{1}{3}i + j - 2k$

$\dfrac{\partial w}{\partial r} = -\dfrac{4}{3} + 1 - 4 = -\dfrac{13}{3} = \dfrac{\partial w}{\partial s} = -\dfrac{1}{3} - 2 - 2 = -\dfrac{13}{3}$

$\Rightarrow \nabla f(5,3,4) = -\frac{1}{3}i + j - 2k$ ga——

<u>Problem 4</u>

$f(x,y,z) = x^2y^2z^2 \Rightarrow \nabla f = 2xy^2z^2 i + 2yx^2z^2 j + 2zx^2y^2 k$

Let $g(x,y,z) = x^2+y^2+z^2-1 \Rightarrow g(x,y,z)=0$ is level surface $x^2+y^2+z^2=1$

$\nabla g = 2xi + 2yj + 2zk$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2xy^2z^2 i + 2yx^2z^2 j + 2zx^2y^2 = \lambda(2xi + 2yj + 2zk) \\ x^2+y^2+z^2 = 1 \end{cases}$$

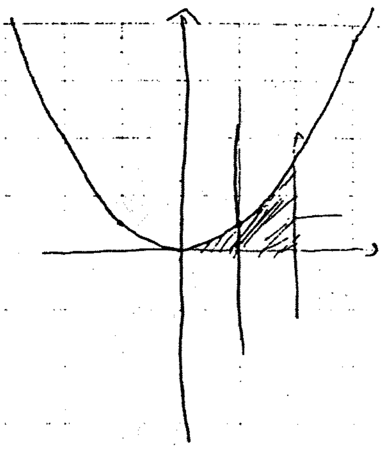
$$\begin{cases} 2xy^2z^2 = \lambda 2x \\ 2yx^2z^2 = \lambda 2y \\ 2zx^2y^2 = \lambda 2z \\ x^2+y^2+z^2=1 \end{cases} \Rightarrow y^2z^2 = x^2z^2 = x^2y^2 \Rightarrow 3x^2 = 1$$

$\Rightarrow y^2 = x^2 = z^2$

$x^2 = \frac{1}{3}$

$x = \pm\frac{\sqrt{3}}{3}$ $y = \pm\frac{\sqrt{3}}{3}$ $z = \pm\sqrt{3}/3$

Exam 2 Fall 01



Problem 1

$$\int_0^2 \int_0^{x^2/2} 2x(1+x^2+y)\,dy\,dx \qquad y=\frac{x^2}{2}, \quad x=\sqrt{2y}$$

$$\int_0^2 \int_{\sqrt{2y}}^{2} 2x(1+x^2+y)\,dx\,dy$$

$$\int_{\sqrt{2y}}^{2} 2x + 2x^3 + 2xy\,dx = x^2 + \frac{2x^4}{2} + x^2y \Big]_{\sqrt{2y}}^{2} = 4 + 8 + 4y - 2y - 2y^2 - 2y^2$$
$$= 12 + 2y - 4y^2$$

$$\int_0^2 (12 + 2y - 4y^2)\,dy = 12y + y^2 - \frac{4y^3}{3}\Big]_0^2 = 72 + 4 - 16 = \frac{52}{3}$$

(b) $$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 e^{r^2} r\,dr\,d\theta$$

$$\int_0^1 e^{r^2} r\,dr = \frac{e^{r^2}}{2}\Big]_0^1 = \frac{e}{2} - \frac{1}{2}$$

$$\Rightarrow I = \int_0^{\pi/2} \left(\frac{e}{2} - \frac{1}{2}\right) d\theta = \frac{e\pi}{4} - \frac{\pi}{4}$$

Problem 2

$z = \ln(2x + 3y + 6z)$

Let $f(x,y,z) = \ln(2x+3y+6z) - z$

$\Rightarrow z = \ln(2x+3y+6z)$ is level surface $f(x,y,z) = 0$

at (2,1) $\dfrac{\partial w}{\partial r} = 4f_x + f_y + 2f_z$

$\dfrac{\partial w}{\partial s} = f_x - 2f_y + f_z$

1st case $\nabla f(5,3,4) = i + 2j + k \Rightarrow \dfrac{\partial w}{\partial r} = 4 + 2 + 2 = 8$

$\dfrac{\partial w}{\partial s} = 1 - 4 + 1 = 2 \Rightarrow$ not equal.

at $\nabla f(5,3,4) = -\frac{1}{3}i + j - 2k$

$\dfrac{\partial w}{\partial r} = -\frac{4}{3} + 1 - 4 = -\frac{13}{3} = \dfrac{\partial w}{\partial s} = -\frac{1}{3} - 2 - 2 = -\frac{13}{3}$

$\Rightarrow \nabla f(5,3,4) = -\frac{1}{3}i + j - 2k$ ga

Problem 4

$f(x,y,z) = x^2y^2z^2 \Rightarrow \nabla f = 2xy^2z^2 i + 2yx^2z^2 j + 2zx^2y^2 k$

Let $g(x,y,z) = x^2+y^2+z^2-1 \Rightarrow g(x,y,z)=0$ is level surface $x^2+y^2+z^2=1$

$\nabla g = 2xi + 2yj + 2zk$

$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2xy^2z^2 i + 2yx^2z^2 j + 2zx^2y^2 = \lambda(2xi + 2yj + 2zk) \\ x^2+y^2+z^2 = 1 \end{cases}$

$\begin{cases} 2xy^2z^2 = \lambda 2x \\ 2yx^2z^2 = \lambda 2y \\ 2zx^2y^2 = \lambda 2z \\ x^2+y^2+z^2 = 1 \end{cases} \Rightarrow y^2z^2 = x^2z^2 = x^2y^2 \Rightarrow 3x^2 = 1$

$\Rightarrow y^2 = x^2 = z^2$

$x^2 = \frac{1}{3}$

$x = \pm\frac{\sqrt{3}}{3}$ $y = \pm\frac{\sqrt{3}}{3}$

$z = \pm\sqrt{3}/3$

Fall 01

(a) $x = r^2 + s \qquad y = \frac{r}{s} + 1 \qquad z = 2r + \ln s \qquad w = f(x, y, z)$

$$\frac{\partial w}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} + f_z \frac{\partial z}{\partial r}$$

$$= f_x(2r) + f_y\left(\frac{1}{s}\right) + f_z(2)$$

$$= 2r(f_x) + \frac{f_y}{s} + 2f_z$$

$$\frac{\partial w}{\partial s} = f_x - \frac{r f_y}{s^2} + \frac{f_z}{s}$$

(b) $(r,s) = (2,1) \Rightarrow \frac{\partial w}{\partial r} = 4f_x + f_y + 2f_z$

$+ \quad \frac{\partial w}{\partial s} = f_x - 2f_y + f_z$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial r}$

$\Rightarrow 3f_x + 3f_y + f_z = 0$

for $r = 2$ + $s = 1$ $\quad x = 5 \quad y = 3 \quad z = 4$

$\Rightarrow \nabla f(2,1,0) = i + j - 6k$. is omitted from choices.

$\nabla f(5,3,4) = i + 2j + k \Rightarrow f_x = 1 \quad f_y = 2 \quad f_z = 1$

but $3f_x + 3f_y + f_z = 0$

$3 + 6 + 1 \neq 0 \Rightarrow \nabla f(5,3,4) = i + 2j + k$ does not satisfy the relation.

$\nabla f(5,3,4) = -\frac{1}{3} i + j - 2k \Rightarrow f_x = -\frac{1}{3} \quad f_y = 1 \quad f_z = -2$

$\Rightarrow 3f_x + 3f_y + f_z = -1 + 3 - 2 = 0 \Rightarrow$ satisfies relation

$\Rightarrow \nabla f(5,3,4)$ is the choice which guarantees the given cond.