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Quiz 2: MATH 201

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Duration: 60 minutes

Name (Last, First):

Student number:

[15 points: 3 points on each part] **Problem 1.** Choose the right answer in each of the following parts (you may use the next sheet as a scratch paper).

- The slope dy/dx of the tangent to the polar curve $r = \sin(2\theta)$ at the point where $\theta = \frac{\pi}{4}$ is

  1. 1
  2. -1
  3. 1/2
  4. 0

- The equivalent of the cartesian equation $x^4 + y^4 + 2x^2y^2 - y^2 - 2xy^2 - 2x^3 = 0$ in polar coordinates is

  1. $r = 1 + \cos\theta$
  2. $r = 1 - \cos\theta$
  3. $r = 1 - \sin\theta$
  4. $r = 1 + \sin\theta$

- Given the function $f(x, y) = \sqrt{\frac{y+x}{y-x}}$. Then the domain of f is a region in the $xy-$plane which is

  1. open and closed
  2. open but not closed
  3. not open but closed
  4. not open and not closed

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- The intersection of the surfaces $2x^2 + y^2 + \frac{z^2}{4} = 24$ and $z = -2$ is

  1. an elliptic cylinder
  2. a circle
  3. an ellipsoid
  4. an ellipse

- The range of the function $f(x, y, z) = \frac{1}{\sqrt{25 - (x^2 + y^2 + z^2)}}$ is

  1. $(5, \infty)$
  2. $(0.2, +\infty)$
  3. the region within a sphere of radius 5
  4. $[0.2, \infty)$

[15 points] **Problem 2.** Use standard linear approximation to estimate $\sqrt{(3.90)^2 + (3.02)^2}$.

Let $f(x,y) = \sqrt{x^2+y^2}$ (where $f(3.9, 3.02) = \sqrt{3.9^2+3.02^2}$)

- $f(4,3) = \sqrt{4^2+3^2} = 5$
- $f_x = \dfrac{2x}{2\sqrt{x^2+y^2}} \Rightarrow f_x\big|_{(4,3)} = \dfrac{\cancel{2}(4)}{\cancel{2}\sqrt{4^2+3^2}} = \dfrac{4}{5}$

  $= \dfrac{x}{\sqrt{x^2+y^2}}$

  $f_y = \dfrac{y}{\sqrt{x^2+y^2}} \Rightarrow f_y\big|_{(4,3)} = \dfrac{3}{\sqrt{4^2+3^2}} = \dfrac{3}{5}$

- Applying the Linear Approximation to $f(x)$:

$L(x,y) = f(4,3) + f_x|_{(4,3)}(x-4) + f_y|_{(4,3)}(y-3)$

$= \cancel{5} + \frac{4}{5}x - \cancel{\frac{16}{5}} + \frac{3}{5}y - \cancel{\frac{9}{5}}$

$= \frac{4}{5}x + \frac{3}{5}y$

So, $\sqrt{3.9^2+3.02^2} = f(3.9, 3.02) \approx L(3.9, 3.02) = 4.932$

$\Rightarrow$ The Estimation of $\sqrt{3.9^2+3.02^2}$ is 4.932.

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[20 points = 10+10] **Problem 3.** Consider the surface $\frac{x^2}{2} + \frac{y^2}{5} = 1 + z^2$.

(a) Find an equation of the tangent plane to the above surface at the point $(2, 0, 1)$.

LET $f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{5} - 1 - z^2$

$$\frac{\partial f}{\partial x} = x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(2,0,1)} = 2$$

$$\frac{\partial f}{\partial y} = \frac{2}{5}y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(2,0,1)} = 0$$

$$\frac{\partial f}{\partial z} = -2z \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{(2,0,1)} = -2$$

$\Rightarrow$ EQ of tangent plane: $\left.\frac{\partial f}{\partial x}\right|_{(2,0,1)}(x-2) + \left.\frac{\partial f}{\partial y}\right|_{(2,0,1)}(y-0) + \left.\frac{\partial f}{\partial z}\right|_{(2,0,1)}(z-1) = 0$

$$2x - 4 + 0 - 2z + 2 = 0$$

$$\Rightarrow 2x - 2z - 2 = 0$$

$$\Rightarrow \boxed{x - z = 1}$$

(b) Find an equation of the normal line to the above surface at the point $(2, 0, 1)$.

EQ of Normal Line: $\begin{cases} x = 2 + \left.\frac{\partial f}{\partial x}\right|_{(2,0,1)} k \\ y = 0 + \left.\frac{\partial f}{\partial y}\right|_{(2,0,1)} k \\ z = 1 + \left.\frac{\partial f}{\partial z}\right|_{(2,0,1)} k \end{cases}$ (where $k \in \mathbb{R}$)

$$\Rightarrow \begin{cases} x = 2 + 2k \\ y = 0 \\ z = 1 - 2k \end{cases}$$

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[20 points=10 +10] Problem 4.

(a) Let

$$f(x,y) = \begin{cases} \dfrac{\cos(x^2+y^2)-1}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ a & \text{if } (x,y) = (0,0). \end{cases}$$

Find the value of a so that the function f is continuous at $(0,0)$.

since f is continuous at $(0,0)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = a$$

$$f(x,y) = \frac{\cos(x^2+y^2)-1}{x^2+y^2}$$

We know that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$$\cos(x^2+y^2) - 1 = -\frac{(x^2+y^2)^2}{2!} + \frac{(x^2+y^2)^4}{4!} - \frac{(x^2+y^2)^6}{6!} + \cdots$$

$$\frac{\cos(x^2+y^2)-1}{x^2+y^2} = -\frac{(x^2+y^2)}{2!} + \frac{(x^2+y^2)^3}{4!} - \frac{(x^2+y^2)^5}{6!} + \cdots$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[-\frac{(x^2+y^2)}{2!} + \frac{(x^2+y^2)^3}{4!} - \frac{(x^2+y^2)^5}{6!} + \cdots \right] = 0$$

So, $a = 0$

(b) (Independent of part (a)) Compute $\lim_{x\to 0} \dfrac{(\sin x - x)(\cos(x^2) - 1)}{x^7}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \cos(x^2) - 1 = -\frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

$$(\sin x - x)(\cos(x^2) - 1) = \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\cdot\left(\frac{-x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right)$$

$$= \frac{x^7}{3!2!} - \frac{x^9}{5!2!} + \left(-\frac{1}{3!4!} + \frac{1}{7!2!}\right)x^{11} + \cdots$$

$$\Rightarrow \frac{(\sin x - x)(\cos(x^2) - 1)}{x^7} = \frac{1}{3!2!} - \frac{x^2}{5!2!} + \left(-\frac{1}{3!4!} + \frac{1}{7!2!}\right)x^4 + \cdots$$

$$\lim_{x\to 0} \frac{(\sin x - x)(\cos(x^2) - 1)}{x^7} = \frac{1}{3!2!} = \frac{1}{12}$$

[20 points=10 + 10] **Problem 5.** Find the limit in each of the parts below, if it exists; or justify why it does not exist.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6\sin(x^2+y^6)}{(x^2+y^6)^2},$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6\sin(x^2+y^6)}{(x^2+y^6)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2y^6}{x^4+y^{12}+2x^2y^6}\cdot \sin(x^2+y^6)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2y^6}{\frac{1}{2}x^4+\frac{1}{2}y^{12}+x^2y^6}\cdot \frac{1}{2}\sin(x^2+y^6)$$

$$\left|\frac{x^2y^6}{\frac{1}{2}x^4+\frac{1}{2}y^{12}+x^2y^6}\cdot \frac{1}{2}\sin(x^2+y^6)\right| = \frac{x^2y^6}{\frac{1}{2}x^4+\frac{1}{2}y^{12}+x^2y^6}\left|\frac{1}{2}\sin(x^2+y^6)\right|$$

$$\Rightarrow \frac{x^2y^6}{\frac{1}{2}x^4+\frac{1}{2}y^{12}+x^2y^6}\left|\frac{1}{2}\sin(x^2+y^6)\right| \leq \left|\frac{1}{2}\sin(x^2+y^6)\right|$$

since $\lim_{(x,y)\to(0,0)} \left|\frac{1}{2}\sin(x^2+y^6)\right| = 0$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y^6\sin(x^2+y^6)}{(x^2+y^6)^2} = 0$$ By Corollary to Sandwich Theorem.

(b) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$

$$\left|\frac{xy^4}{x^4+y^4}\right| \leqslant \left|\frac{x(x^4+y^4)}{x^4+y^4}\right| = |x|$$

since $\lim_{(x,y)\to(0,0)} |x| = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = 0$ By corollary to Sandwich theorem

[10 points] Problem 6. Let f be a function of two variables that satisfies $f(tx, ty) = t^3 f(x, y)$ for all $t \in \mathbb{R}$ and f has continuous second-order partial derivatives. Show that

$$x^2 \frac{\partial^2 f}{\partial x^2}(x, y) + 2xy \frac{\partial^2 f}{\partial x \partial y}(x, y) + y^2 \frac{\partial^2 f}{\partial y^2}(x, y) = 6f(x, y).$$

Hint: use the chain rule to differentiate the function $h(t) = f(tx, ty)$ with respect to t.

Let $u = tx$ & $v = ty$

$$h(t) = f(tx, ty) = f(u,v) \Rightarrow \frac{dh}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} = \frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v} y$$

& $h(t) = t^3 f(x,y) \Rightarrow \dfrac{dh}{dt} = 3t^2 f(x,y)$

So, $\dfrac{\partial f}{\partial u} x + \dfrac{\partial f}{\partial v} y = 3t^2 f(x,y)$

Differentiate again both sides wrt t:

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v} y\right) = 6t\, f(x,y)$$

- $$\frac{\partial}{\partial t}\left(x \frac{\partial f}{\partial u}\right) = x\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial t} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial t}\right) = x\left(\frac{\partial^2 f}{\partial u^2} x + \frac{\partial^2 f}{\partial u \partial v} y\right) = x^2 \frac{\partial^2 f}{\partial u^2} + xy \frac{\partial^2 f}{\partial u \partial v}$$

- similarly, $\dfrac{\partial}{\partial t}\left(y \dfrac{\partial f}{\partial v}\right) = y^2 \dfrac{\partial^2 f}{\partial v^2} + xy \dfrac{\partial^2 f}{\partial u \partial v}$

$$\Rightarrow x^2 \frac{\partial^2 f}{\partial u^2} + 2xy \frac{\partial^2 f}{\partial u \partial v} + y^2 \frac{\partial^2 f}{\partial v^2} = 6t\, f(x,y)$$

<u>FOR $t=1$:</u> $u = x$ & $v = y$

$$\text{So, } x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 6 f(x,y)$$