Exercise 1. Let $f(x,y) = \frac{1}{\sqrt{9-x^2-y^2}}$.

(a) **(8 points)** Find the domain D of f. Is D open, closed? Is D bounded or unbounded ?

$9-x^2-y^2 \geq 0$ and $\sqrt{9-x^2-y^2} \neq 0 \Rightarrow$

$9-x^2-y^2 > 0 \Rightarrow \boxed{x^2+y^2 < 9}$ so D is the circle of center $(0,0)$ and radius 3

D is open, not closed, bounded.

(b) **(5 points)** Find the range of f.

Let $c \in$ range of f $\Rightarrow c = \frac{1}{\sqrt{9-x^2-y^2}} \Rightarrow (c>0)$

$c^2 = \frac{1}{9-x^2-y^2} \Rightarrow 9-x^2-y^2 = \frac{1}{c^2} \Rightarrow \underbrace{x^2+y^2}_{>0} = 9 - \frac{1}{c^2} \Rightarrow$

$9 - \frac{1}{c^2} > 0 \Rightarrow \frac{1}{c^2} < 9 \Rightarrow c^2 > \frac{1}{9} \Rightarrow \left(\underset{\text{rej}}{c < -\frac{1}{3}}\right)$ or $c > \frac{1}{3}$

so $c \in \left(\frac{1}{3}, +\infty\right)$ range of f

(c) **(7 points)** Describe the level curves of f. Determine the level curve of f passing through $(1, 2)$.

Let $\frac{1}{\sqrt{1-x^2-y^2}} = c \Rightarrow 1-x^2-y^2 = \frac{1}{c^2} \Rightarrow x^2+y^2 = 1-\frac{1}{c^2}$

so the level curves are semi-~~spheres~~ circles with center $(0,0)$ and radius $\sqrt{1-\frac{1}{c^2}}$

$(1,2) \in f(x,y) \Rightarrow f(1,2) = \frac{1}{\sqrt{1-1-4}} = \boxed{\frac{1}{2}} \Rightarrow \boxed{c = \frac{1}{2}} \Rightarrow \boxed{x^2+y^2=5}$

so it is a semi-sphere of center $(0,0)$ and radius $\sqrt{5}$

Exercise 2. Determine the following limits if they exist. Justify your answer.

(a) **(8 points)** $\lim\limits_{(x,y)\to(0,0)} \dfrac{5y^4x}{2x^4+3y^4}$

$$\left|\frac{5y^4x}{3x^4+3y^4}\right| \leq \left|\frac{5y^4\,x}{3y^4}\right| = \frac{5}{3}|x| \Longrightarrow \left|\frac{3y^4x}{3x^4+3y^4}\right| \leq \frac{5}{3}|x|$$

so by CST : $\lim\limits_{(x,y)\to 0} \dfrac{3y^4x}{3x^4+3y^4} = 0$ $\left(\text{because } \lim\limits_{(x,y)\to(0,0)} \dfrac{5}{3}|x| = 0\right)$

15

(b) **(8 points)** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5y}{x^{10}+y^2}$

$m>1$

Let $x^{10}+y^2 = mx^{\alpha} \Rightarrow y^2 = mx^{\alpha} - x^{10}$

so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5\cdot\sqrt{mx^{\alpha}-x^{10}}}{mx^{\alpha}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^{10}\sqrt{mx^{\alpha-10}-1}}{mx^{\alpha}}$

Let $\alpha = 10$ so path test with $y = mx^{10} - x^{10}$

we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{mx^0-1}}{m} = \boxed{\dfrac{\sqrt{m-1}}{m}}$

this is dependent on m, so different values of give different values of limit $\Rightarrow$ limit does not exist

Exercise 3. Consider the following function:

$$f(x,y)=\begin{cases} \dfrac{y^2}{x} & \text{for } x\neq 0\\ y & \text{for } x=0\end{cases}$$

(a) **(8 points)** Using the definition of partial derivatives, determine $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

using definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim \frac{\frac{0^2}{h}-0}{h} = \frac{0}{6}$$

8

$$\frac{\partial f}{\partial y}(0,0) = \lim_{K\to 0}\frac{f(0,0+K)-f(0,0)}{K} = \lim_{K\to 0}\frac{K-0}{K} = \boxed{1}$$

(b) **(7 points)** Is f continuous at $(0,0)$? Justify you answer.

(i) $(0,0) \in$ domain of f

(ii) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^2}{x}$

path test take $y = m x^{\frac{1}{2}}$

so $\lim_{(x,y)\to(0,0)} \frac{m^2 x}{x} = \boxed{m^2}$ this is dependent on m

so limit does not exist

f is not continuous on $(0,0)$

7

(c) **(5 points)** Deduce whether or not the function f is differentiable at $(0,0)$.

since f is not continuous on $(0,0) \Rightarrow f$ is not differentiable on $(0,0)$

5

xercise 4. Consider the surface S defined by the equation

$$5x^2 + 3e^y = 5 + 3\ln z.$$

a) **(8 points)** Find the equation of the tangent plane to S at the point $(1, 0, e)$.

let $f(x,y,z) = 5x^2 + 3e^y - 5 - 3\ln z = 0$ level surface

$\vec{\nabla} f(x,y,z) = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = 10x\vec{i} + 3e^y\vec{j} - \frac{3}{z}\vec{k}$

$\vec{\nabla} f(1,0,e) = 10\vec{i} + 3\vec{j} - \frac{3}{e}\vec{k}$

$\vec{\nabla} f(1,0,e)$ is normal to the plane and $(1,0,e) \in$ plane

so the eq is: $10(x-1) + 3(y-0) - \frac{3}{e}(z-e) = 0 \Rightarrow$

$$\boxed{10x + 3y - \frac{3}{e}z - 7 = 0}$$

(b) **(7 points)** Find the normal line to S at the point $(1, 0, e)$.

$\vec{\nabla} f(1,0,e)$ is a direction vector to the normal line to S at $(1,0,e)$ and $(1,0,e) \in$ normal

so the normal is $\begin{cases} x = 10t + 1 \\ y = 3t \\ z = -\frac{3}{e}t + e \end{cases}$ t real parameter

Exercise 5. (9 points) Find $\frac{\partial z}{\partial x}$ at $(0,1)$ if $z = f(x,y)$ is determined implicitly by the equation

$$yx^2 + z + \cos(xyz) - 5 = 0.$$

by implicit differentiation: Let $F(x,y,z) = yx^2 + z + \cos(xyz) - 5 = 0$

$$\frac{\partial z}{\partial x}(0,1) = -\frac{F_x}{F_z}(0,1)$$

- $F_x = 2xy - yz\sin(xyz)$
- $F_z = 1 - xy\sin(xyz)$

so $$\frac{\partial z}{\partial x}(0,1) = -\frac{2xy - yz\sin(xyz)}{1 - xy\sin(xyz)}(0,1)$$

$$= -\frac{0}{1} = \boxed{0}$$

9

Exercise 6. Let $f(x, y)$ be a differentiable function of two variables. Suppose that

$$f(0,3) = 9, \quad f(-1,5) = 2,$$

and

$$\nabla f(0,3) = 2\mathbf{i} - 3\mathbf{j}, \qquad \nabla f(-1,5) = 6\mathbf{i} + \mathbf{j}.$$

Finally let

$$x = r - 2s, \quad y = r + s,$$

and

$$w = f(x, y).$$

(a) **(15 points)** Find the partial derivatives $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ at $(r, s) = (2, 1)$. Then estimate w at the point $(r, s) = (1.97, 1.04)$.

by chain rule II:

$$* \quad \frac{\partial w}{\partial r} = \left[\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}\right] \cancel{+ \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}}$$

$$= f_x \cdot 1 + f_y \cdot 1 = f_x + f_y$$

- if $\left.\begin{matrix} r = 2 \\ \text{and } s = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} x = 0 \\ y = 3 \end{matrix}$

we have $\vec{\nabla} f(0,3) = 2\vec{i} - 3\vec{j} \Rightarrow f_x(0,3) = 2$ and $f_y(0,3) = -3$

then $\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(2,1)} = f_x(0,3) + f_y(0,3) = 2 - 3 = \boxed{-1}$

15

- $$\frac{\partial w}{\partial s} = \left[\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}\right]$$

$$= -2f_x + f_y$$

$$\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(2,1)} = -2f_x(0,3) + f_y(0,3) = -4 - 3 = \boxed{-7}$$

$$* \quad D_u w = \frac{\Delta w}{\Delta t} \Rightarrow \Delta w = D_u w \cdot \Delta t = \vec{\nabla} w_{(2,1)} \vec{u} \cdot \Delta t$$

- $\vec{\Delta w}(2,1) = (w_r \vec{i} + w_s \vec{j})_{(2,1)} = -\vec{i} - 7\vec{j}$

$$\vec{u} = \frac{(1.97-2)\vec{i} + (1.04-1)\vec{j}}{\sqrt{0.03^2 + 0.04^2}} = \frac{-0.03\vec{i} + 0.04\vec{j}}{\sqrt{0.03^2 + 0.04^2}} \; ; \; \Delta t = \sqrt{0.03^2 + 0.04^2}$$

so $\Delta w = (-\vec{i} - 7\vec{j}) \cdot (-0.03\vec{i} + 0.04\vec{j}) = 0.03 - 0.28 = \boxed{-0.25}$

$= -0.25 \Rightarrow w(1.97, 1.04) = -0.25 + w(2,1)$

$\Rightarrow w(1.97, 1.04) - w(2,1) = -0.25$

$$w(2,1) = f(0,3) = 9 \Longrightarrow$$

$$W(1.97, 1.04) = 9 - 0.25 = \boxed{8.75}$$

(b) (5 points) Find the normal line to the surface $z = f(x, y)$ at the point $(-1, 5, 2)$.

Let $h(x,y,z) = f(x,y) - z = 0$ level surface $(\underbrace{f(-1,5)}_{2} - z = 0)$

$\vec{\nabla} h_{(-1,5,2)} = \left[h_x \vec{i} + h_y \vec{j} + h_z \vec{k} \right]_{(-1,5,2)}$

$h_x = \frac{\partial}{\partial x}(f(x,y) - z) = \frac{\partial f(x,y)}{\partial x} \Rightarrow h_x = f_x(-1,5) = \boxed{6}$ $\left(\vec{\nabla} f(-1,5) = \underbrace{6\vec{i}}_{f_x(-1,5)} + \vec{j}\right)$

$h_y = \frac{\partial}{\partial y}(f(x,y) - z) = \frac{\partial f(x,y)}{\partial y} \Rightarrow h_y = f_y(-1,5) = \boxed{1}$

$h_z = \frac{\partial}{\partial z}(f(x,y) - z) = \boxed{-1}$ ($f(x,y)$ independent of z)

so $\boxed{\vec{\nabla} h_{(-1,5,2)} = 6\vec{i} + \vec{j} - \vec{k}}$

normal line (d):

$\vec{\nabla} h(-1,5,2)$ is a direction ~~of derivative~~ vector to (d) and

$(-1,5,2) \in (d) \Longrightarrow (d) \begin{cases} x = 6m - 1 \\ y = m + 5 \\ z = -m + 2 \end{cases}$ m real parameter

5