

Exercise 6 (15 pts) Solve the DE: $(3x + 1/2)y^2 \frac{dy}{dx} + \frac{3}{2x^2} + y^3 = 0$

Solution: Divide by y^2 :

$$(3x + \frac{1}{2}) \frac{dy}{dx} + \frac{3}{2x^2 y^2} + \frac{y^3}{y^2} = 0$$

$$(3x + \frac{1}{2}) y' + y = \frac{3}{2x^2} y^{-2} \Rightarrow y' + \frac{1}{(3x + \frac{1}{2})} y = \frac{3}{2x^2 (3x + \frac{1}{2})} y^{-2}$$

Bernoulli with $n = -2$

Both sides
Multiply y^2 \rightarrow $y^2 y' + \frac{1}{(3x + \frac{1}{2})} y^3 = \frac{3}{2x^2 (3x + \frac{1}{2})}$

$u = y^3 = y^{1-n} = y^{1-(-2)} = y^3$
 $u' = 3y^2 y'$
 $\frac{1}{3} u' = y^2 y'$

Replace $\rightarrow \frac{1}{3} u' + \frac{1}{(3x + \frac{1}{2})} u = \frac{3}{2x^2 (3x + \frac{1}{2})}$
 $\rightarrow u' + \frac{3}{(3x + \frac{1}{2})} u = \frac{9}{2x^2 (3x + \frac{1}{2})}$
 $I(x) = e^{\int \frac{3}{3x + \frac{1}{2}} dx} = e^{\ln(3x + \frac{1}{2})} = (3x + \frac{1}{2})$

$$(3x + \frac{1}{2}) u' + (3x + \frac{1}{2}) \cdot \frac{3}{(3x + \frac{1}{2})} u = (3x + \frac{1}{2}) \cdot \frac{9}{2x^2 (3x + \frac{1}{2})}$$

$$(3x + \frac{1}{2}) u' + 3u = \frac{9}{2} x^{-2} \Rightarrow \left[(3x + \frac{1}{2}) u \right]' = \frac{9}{2} x^{-2}$$

$$\Rightarrow (3x + \frac{1}{2}) u = \int \frac{9}{2} x^{-2} dx = -\frac{9}{2} x^{-1} + C$$

$$\Rightarrow \boxed{(3x + \frac{1}{2}) y^3 = -\frac{9}{2} x^{-1} + C}$$

Exercise 7 (15 pts) Multiply both sides of the differential equation by an appropriate factor to make it exact. Then Solve it: $(3xy^2 + 2x^2y)dy + (2y^3 + 3xy^2)dx = 0$

Solution: First divide by y : $(3xy + 2x^2)dy + (2y^2 + 3xy)dx = 0$

$$u = \frac{y}{x}$$

Homogeneous of degree $\frac{2}{1}$

$$\boxed{y = ux} \Rightarrow \boxed{dy = udx + xdu}$$

$$(3x(ux) + 2x^2)(udx + xdu) + (2(ux)^2 + 3(x)(ux))dx = 0$$

$$\Rightarrow [3x^2u + 2x^3]du + [3x^2u^2 + 2x^2u + 2u^2x^2 + 3x^2u]dx = 0$$

$$\xrightarrow{\text{Divide } x^2} [3xu + 2x]du + [3u^2 + 2u + 2u^2 + 3u]dx = 0$$

$$\Rightarrow x[3u + 2]du + 5[u^2 + u]dx = 0$$

$$\Rightarrow \frac{3u+2}{5(u^2+u)} du + \frac{dx}{x} = 0 \Rightarrow \frac{3u+2}{5(u^2+u)} du = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{5} \int \frac{3u+2}{u^2+u} du = \int -\frac{dx}{x} \Rightarrow \frac{1}{5} \int \left(\frac{2}{u} + \frac{1}{u+1} \right) du = -\ln|x| + C$$

$$\Rightarrow \frac{2}{5} \ln|u| + \frac{1}{5} \ln|u+1| = -\ln|x| + C$$

$$\Rightarrow \boxed{\frac{2}{5} \ln\left(\frac{y}{x}\right) + \frac{1}{5} \ln\left(\frac{y}{x} + 1\right) = -\ln|x| + C}$$

$$\frac{3u+2}{u^2+u} = \frac{3u+2}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} = \frac{Au + A + Bu}{u(u+1)} = \frac{(A+B)u + A}{u(u+1)}$$

$$\Rightarrow \begin{cases} A+B=3 \\ A=2 \end{cases} \Rightarrow \boxed{B=1} \Rightarrow \boxed{A=2} \Rightarrow \frac{3u+2}{u^2+u} = \frac{2}{u} + \frac{1}{u+1}$$