

Exercise 1 (15 pts) Use Divergence Theorem to find the outward flux

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

of the vector field

$$\vec{F} = 5x \vec{i} + 2y \vec{j} + 8z \vec{k}$$

across the sphere $\rho = 2$

Solution : $\vec{F} = 5x \vec{i} + 2y \vec{j} + 8z \vec{k} \Rightarrow$ so \vec{F} is continuous

and no bad points. and $\text{div}(\vec{F}) = \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(8z) = 5 + 2 + 8 = 15$

By Divergence Theorem, $\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_V \text{div}(\vec{F}) \, dV$

$$= \iiint_V 15 \, dV$$

$$= 15 (\text{Volume})$$

$$= 15 \left(\frac{4}{3} \pi (2)^3 \right)$$

$$= 160 \pi.$$

Exercise 2 (15 pts) Let $\vec{F} = 5y\vec{i} + 8x\vec{j} + z\vec{k}$ and S be the paraboloid

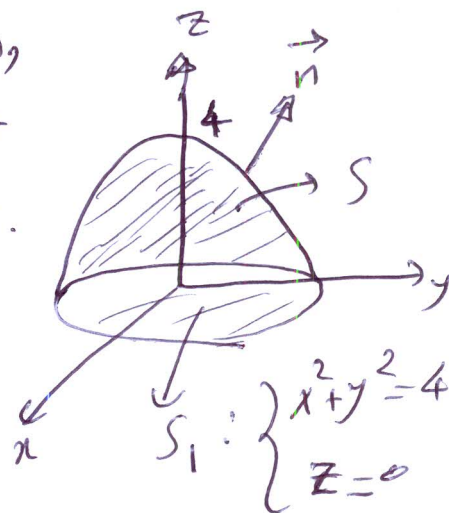
$$z = 4 - x^2 - y^2, z \geq 0 \rightarrow \text{div}(\vec{F}) = 0 + 0 + 1 = 1$$

which is open at the bottom. Find

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

(where \vec{n} is the outer normal vector to our surface) by using Divergence Theorem

Solution: Since S is open at the bottom, we add S_1 ($z=0, x^2+y^2=4$) to make it closed and to get balance, subtract it.

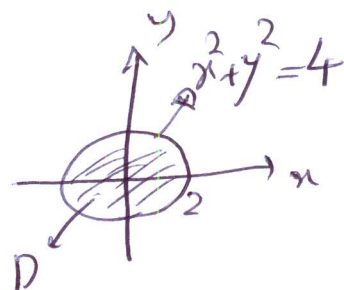


$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \left[\iint_S \vec{F} \cdot \vec{n} \, d\sigma + \iint_{S_1} \vec{F} \cdot \vec{n} \, d\sigma \right] - \iint_{S_1} \vec{F} \cdot \vec{n} \, d\sigma$$

the normal vector for S_1 is $\vec{n} = -\vec{k}$

$$= \iint_{S \cup S_1} \vec{F} \cdot \vec{n} \, d\sigma - \iint_{S_1} \vec{F} \cdot \vec{n} \, d\sigma$$

$S \cup S_1$ is closed, so we can use Divergence Thm



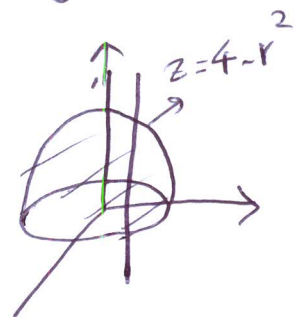
$$= \iiint_V \text{div}(\vec{F}) \, dV - \iint_D (5y\vec{i} + 8x\vec{j} + z\vec{k}) \cdot (-\vec{k}) \, dx \, dy$$

$$= \iiint_V dV - \iint_D z \, dx \, dy$$

$\int_{x^2+y^2=4}^{z=0}$

$$= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4-r^2)r \, dr \, d\theta = \dots$$



Exercise 3 Let S be the upper spherical cap formed by cutting the sphere $x^2 + y^2 + z^2 = 2$ with a cone having the equation $z = \sqrt{x^2 + y^2}$. Answer the following questions:

a) (10 pts) Let C denote the boundary of the surface (the cap) S and let

$$\vec{F} = (z - y)\vec{i} + y\vec{k}$$

Calculate the circulation of \vec{F} around the curve C counter clockwise by parameterizing the curve C

Solution :

$C =$ the circle ($z=1, x^2+y^2=1$)

$=$ the circle with radius 1 and center $(0,0,1)$

$$\Rightarrow C: \begin{cases} x = \cos \theta \Rightarrow dx = -\sin \theta d\theta \\ y = \sin \theta, \quad 0 \leq \theta \leq 2\pi \\ z = 1 \Rightarrow dz = 0 \end{cases}$$

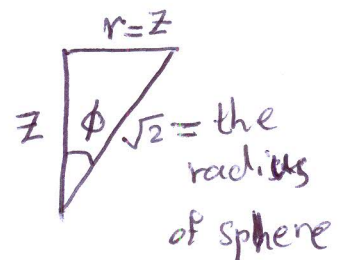
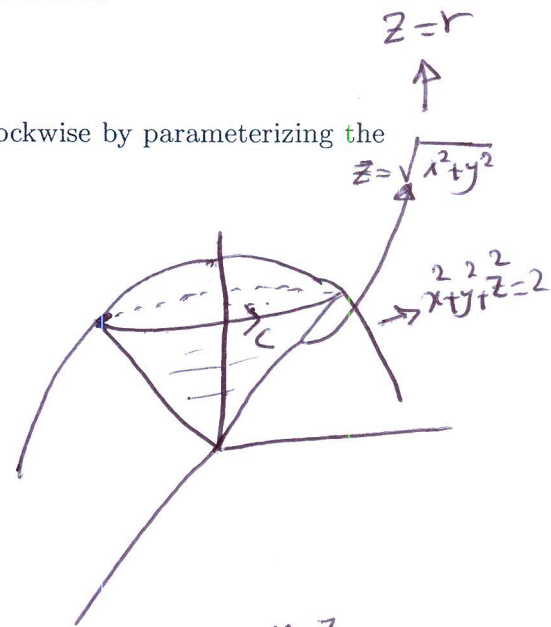
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M dx + N dy + P dz$$

$$= \int_0^{2\pi} (z-y) dx + 0 dy + y dz$$

$$= \int_0^{2\pi} (1 - \sin \theta)(-\sin \theta) d\theta = \int_0^{2\pi} (\sin \theta + \sin^2 \theta) d\theta = \int_0^{2\pi} \left(-\sin \theta + \frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= \left[\cos \theta + \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \left[\cos 2\pi + \frac{1}{2}(2\pi) - \frac{1}{4} \sin(2(2\pi)) \right] - \left[\cos 0 + \frac{1}{2}(0) - \frac{1}{4} \sin 0 \right]$$

$$= 1 + \pi - 1 = \pi$$



$$\tan \phi = \frac{r}{z} = \frac{z}{z} = 1$$

$$\Rightarrow \phi = \frac{\pi}{4}$$

$$z^2 + r^2 = (\sqrt{2})^2 = 2$$

$$z^2 + z^2 = 2$$

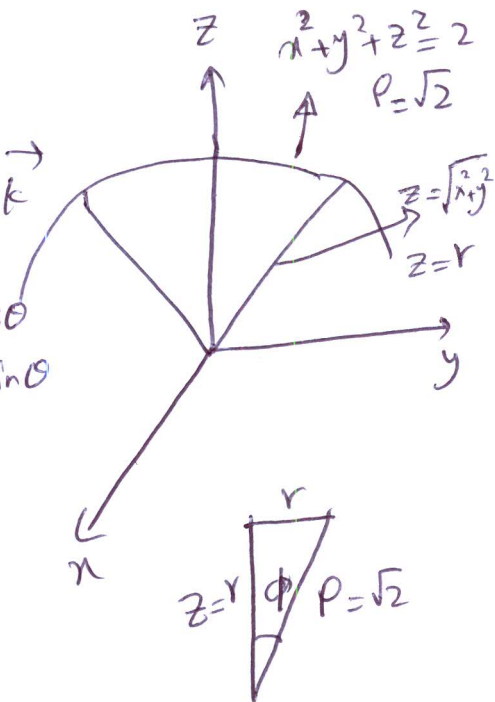
$$\Rightarrow \boxed{z = r = 1}$$

b) (10 pts) Solve part a) by using Stokes' Theorem

$$\vec{S}(\phi, \theta) = \sqrt{2} \sin\phi \cos\theta \vec{i} + \sqrt{2} \sin\phi \sin\theta \vec{j} + \sqrt{2} \cos\phi \vec{k}$$

$0 \leq \phi \leq \pi/4, \quad 0 \leq \theta \leq 2\pi$

by using the spherical coordinates: $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$



$$\int_{\phi} \vec{x} \times \int_{\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2} \cos\phi \cos\theta & \sqrt{2} \cos\phi \sin\theta & -\sqrt{2} \sin\phi \\ -\sqrt{2} \sin\phi \sin\theta & \sqrt{2} \sin\phi \cos\theta & 0 \end{vmatrix} = \begin{matrix} \vec{i} (2 \sin^2 \phi \cos\theta) \\ \vec{j} (2 \sin^2 \phi \sin\theta) \\ \vec{k} (\sin 2\phi) \end{matrix}$$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & 0 & y \end{vmatrix} = \begin{matrix} \vec{i} (1-0) \\ \vec{j} (0-1) \\ \vec{k} (0-(-1)) \end{matrix} = \vec{i} + \vec{j} + \vec{k}$$

$$\tan \phi = \frac{r}{z} = \frac{r}{r} = 1$$

$$\Rightarrow \phi = \pi/4$$

$$\int_S \text{curl}(\vec{F}) \cdot \vec{n} \, d\omega = \int_0^{2\pi} \int_0^{\pi/4} (\vec{i} + \vec{j} + \vec{k}) \cdot (2 \sin^2 \phi \cos\theta \vec{i} + 2 \sin^2 \phi \sin\theta \vec{j} + \sin 2\phi \vec{k}) \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left(2 \sin^2 \phi (\cos\theta + \sin\theta) + \sin 2\phi \right) \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[(\cos\theta + \sin\theta) \left(\phi - \frac{1}{2} \sin 2\phi \right) + \left(-\frac{1}{2} \cos 2\phi \right) \right]_0^{\pi/4} \, d\theta$$

$$= \int_0^{2\pi} \left[(\cos\theta + \sin\theta) \left(\frac{\pi}{4} - \frac{1}{2} \right) + \frac{1}{2} \right] \, d\theta = \left(\frac{\pi}{4} - \frac{1}{2} \right) (\sin\theta - \cos\theta) \Big|_0^{2\pi} + \left(\frac{1}{2} \theta \right) \Big|_0^{2\pi}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2} \right) \left[(0-1) - (1-0) \right] + \pi = \pi$$

Exercise 4 (10 pts) Solve the IVP:

$$x \frac{dy}{dx} - 2y = x^3 + 3, \quad y(1) = 2$$

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 + \frac{3}{x}, \quad y(1) = 2$$

$$I(x) = e^{-\int \frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$x^{-2} \frac{dy}{dx} - 2x^{-3}y = 1 + 3x^{-3}$$

$$(x^{-2}y)' = 1 + 3x^{-3}$$

$$\Rightarrow$$

$$x^{-2}y = 1 - \frac{3}{2}x^{-2} + C$$

$$\xrightarrow[y=2]{x=1} 1^{-2}(2) = 1 - \frac{3}{2}(1)^{-2} + C$$

$$\Rightarrow 2 = 1 - \frac{3}{2} + C$$

$$\Rightarrow \frac{5}{2} = C$$

$$\Rightarrow \boxed{x^{-2}y = 1 - \frac{3}{2}x^{-2} + \frac{5}{2}}$$