MATHEMATICS 202
(1st Semester, 1994-95)
Quiz II

January 5, 1995

1) Solve the initial value problem

$$y'' + 2y' + 2y = -2\cos 2x - 4\sin 2x, \qquad y(0) = 1,\ y'(0) = 1.$$

2) Find the general solution of the equation

$$x^2 y'' + xy' - y = x^3 e^x$$

3) Find all values of m for which the problem

$$y'' + my = 0, \qquad y(0) = 0, \qquad y'(1) = 0$$

has non-trivial solutions and find those solutions .

4) Let $y = \sum_{n=0}^{\infty} c_n x^n$ be a solution of the equation $y' - 4x^3 y = 0$ satisfying $y(0) = 1$. Find c_n as a function of n and then find y .

5) Let y_1 & y_2 be solutions of the equation :

$$y'' + p(x)y' + q(x)y = 0$$

and let w be the wronskian of y_1 & y_2. Prove that

$$w(x) = w(a)\exp\left(-\int_a^x p(t)\,dt\right), \qquad (\exp t = e^t).$$

Solution
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1) $y'' + 2y' + 2y = -2\cos 2x - 4\sin 2x$, $y(0) = 1$, $y'(0) = 1$.

$m2 + 2m + 2 = 0$

$m1 = -1 - i$, $m2 = -1 + i$

$y_c = e^{-x}(c_1 \cos x + c_2 \sin x)$

Hint : yp is found by the method of undetermined coefficients, don't use the variation of parameters method or else you will face difficult integrations .

2) $x^2 y'' + xy' - y = x^3 e^x$

$m(m-1) + m - 1 = 0$ => $m_1 = 1$, $m_2 = -1$

$y_c = c_1 x^{-1} + c_2 x^1$

yp is determined by variation of parameters .

N.B. not by the undetermined coefficients . (x^2, x are not alway const

$y_1 = x^{-1}$, $y_2 = x$

$y_p = u_1 y_1 + u_2 y_2$ $u_1' = -\frac{y_1 (x^3) e^x}{W}$

Find u_1 & u_2. $u_1' = -\frac{y_2 x^3 e^x}{W}$

3) $y'' + my = 0$, $y(0) = 0$, $y'(1) = 0$

Suppose $m = k$

=> $m^2 + k = 0$

case 1:

$k = 0$

=> $m2 = 0$. Therefore, $y = c_1 + c_2 x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y'(1) = 0 \Rightarrow c_2 = 0$

=> no non-trivial solution .

case 2:

$$k < 0 \quad => \quad k = |k|$$

$$m^2 + |k| = 0$$

$$\Delta = -4|k| = 4|k|i^2$$

$$m_1, m_2 = \pm\frac{2\sqrt{|k|}\ \mathrm{i}}{2} = \pm\sqrt{|k|}\ \mathrm{i}$$

$$y = c_1 e^{-|k|x} + c_2 e^{|k|x}$$

$$y' = -|k|c_1 e^{-|k|x} + |k|c_2 e^{|k|x}$$

$$y(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$$

$$y'(1) = 0 \quad \Rightarrow \quad -|k|c_1 e^{-|k|} + |k|c_2 e^{|k|}$$

$$w = |k|e^{|k|} + |k|e^{-|k|} = 2\cosh e^{|k|} \neq 0$$

$$\Rightarrow c_1 = c_2 = 0$$

=> no non-trivial solution .

case 3 :

$$k > 0 \quad => \quad k = |k|$$

$$\Delta = -4|k| = 4|k|i^2$$

$$m_1, m_2 = \pm\frac{2\sqrt{|k|}\ \mathrm{i}}{2} = \pm\sqrt{|k|}\ \mathrm{i}$$

$$y = c_1 \cos\sqrt{|k|}\ \mathrm{x} + \mathrm{c}_2 \sin\sqrt{|k|}\ \mathrm{x}$$

$$\mathrm{y}(0) = 0 \quad \Rightarrow \quad \mathrm{c}_1 = 0$$

$$y'(1) = 0 \quad \Rightarrow \quad \mathrm{c}_2\sqrt{|\mathrm{k}|}\cos\sqrt{|\mathrm{k}|} = 0$$

c_2 must be $\neq 0$

$$\Rightarrow \quad \sqrt{|\mathrm{k}|}\cos\sqrt{|\mathrm{k}|} = 0$$

$$\sqrt{|\mathrm{k}|} = \mathrm{n}\frac{\pi}{2} \qquad \mathrm{n} = 1/2,\ 3/2,\ 5/2 \ldots$$

$$\Rightarrow \mathrm{y} = \mathrm{c}_2 \sin\left(\frac{n\pi}{2}\right)x$$ is an eigen function

4)

$$y'-4x^3y=0$$

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^{n+3} = 0$$

$$\sum_{k=0}(k+1)c_{k+1}x^k - 4\sum_{k=3} c_{k-3}x^k = 0$$

$$y(0)=1 \quad \Rightarrow \quad c_0 = 1 \qquad c_1 = c_2 = c_3 = 0$$

$$k=3$$

$$c_{k+1} = \frac{4c_{k-3}}{k+1}$$

$$\Rightarrow \quad c_4 = 1 \qquad c_5 = c_6 = c_7 = 0$$

$$c_8 = \frac{4c_4}{8} = 1/2$$

$$c_{12} = 1/12$$

$$c_{4n} = \frac{1}{n!}$$

$$\Rightarrow \quad y = \sum \frac{x^{4n}}{n!} = e^{x^4}$$

5)

$$y'' + p(x)y' + q(x)y = 0$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w' = y_1 y_2' + y_2'' y_1 - y_1' y_2' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''$$

$$w' = y_1(-p(x)y'_2 - q(x)y_2) + y_2(p(x)y'_1 + q(x)y_1)$$

$$w' = -p(x)(y_1 y'_2 - y_2 y'_1)$$

$$w' = -p(x)w$$

$$w' + p(x)w = 0$$

$$\ln w + c = -\int_a^x p(x)dx$$

$$\ln w + c = -\int_a^x p(t)dx$$

$$wx = e^c e^{-\int_a^x p(t)dt}$$

$$\textit{for } x = a \quad \Rightarrow \quad e^c = wa$$