Section 1.1

- *Two types of D.E.*

1. **Ordinary D.E.** ex: $\frac{du}{dx}-\frac{dv}{dx}=x$
2. **Partial D.E.** ex: $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

- The order of the highest order derivative is called the order of the D.E.
- A D.E. is said to be linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

- *Two properties must be noted*

3. The dependent variable y and all its derivatives are of the first degree ex: y''- 2y'+ y = 0 (linear), $(y'')^2$ - 2y'+y = 0 (non linear).
4. Each coefficient depends only on the independent variable x. ex: yy''-2y'=x (non linear)

- An O.D.E. that can be written in the form y = f(x) is said explicit. -A relation G(x,y) = 0 is said to be an implicit solution of an O.D.E. provided it defines one or more explicit solutions on I. ex: For -2< x < 2 $x^2+y^2-4=0$ is on implicit solution of the D.E. $\frac{dy}{dx}=-\frac{x}{y}$. $x^2+y^2-4=0$ has two explicit differentiable functions $y=\pm\sqrt{4-x^2}$ on the interval (-2,2).

Section 2.2

Theorem 2.1: Let R be a rectangular region in the x-y plane defined by a ≤ x ≤ b, c≤ x ≤ d, that contains the point (x_0, y_0) in its interior. If f(x , y) and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_0 and a unique function y(x) defined on I satisfying the initial value problem.

ex : (y - x)y' = y + x ⇒ $y'=\frac{y+x}{y-x}$ ⇒ $f(x,y)=\frac{y+x}{y-x}$ $y\neq x$

2. assume that $\frac{\partial f}{\partial x} = M(x,y)$

3. $f(x, y) = \int M(x,y)dx + g(y)$

4. $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y)$

Section 2.5 : Linear equation of order one.

- A D.E. of the form $a_1(x)\frac{dy}{dx} + a_0 y = g(x)$ is said to be a D.E. of order 1.
- Divide by a1(x) we get $\frac{dy}{dx} + p(x)y = f(x)$
- To solve this D.E. we have to find an integrating factor. $\mu = e^{\int p(x)dx}$ The solution is $y = e^{\int -p(x)dx}\int e^{\int p(x)dx} f(x)dx + Ce^{\int -p(x)dx}$

Section 2.6 :

- Bernouilli's equation $\frac{dy}{dx} + p(x)y = f(x)y^n$
  Devide by $y^n \Rightarrow y^{-n}dy/dx + p(x)y^{1-n} = f(x)$ Let $w = y^{1-n} \Rightarrow dw/dx = (1-n)y^{-n}dy/dx$
  $\Rightarrow \frac{dw}{dx} + (1-n)p(x)w = (1-n)f(x)$ is a linar first order equation.
- Ricatti's equation $\frac{dy}{dx} = p(x) + Q(x)y + R(x)y^2$
  If y_1 is a known particular solution $\Rightarrow y = y_1 + u$ and $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$
  $\Rightarrow \frac{du}{dx} - (a + 2y_1R)u = Ru^2$ ---- which is Bernouilli.
- Claireau equation : $y = xy' + f(y')$
  $$\begin{cases} y = cx + f(c) & \text{general solution} \\ x = -f'(t) & \text{substituting } y = f(t) - tf'(t) \text{ singular solution.} \end{cases}$$

Section 2.7 substitutions.

Section 4.1

Let $a_o(x)$, $a_1(x)$, $a_2(x)$,-----, $a_n(x)$; $g(x_1)$, $g(x_2)$,.....,$g(x_n)$....(A) be continuous on an interval I that contain x_o. Let $a_n(x) \neq 0 \ \forall I \ni x$, then (A) has a unique solution.

ex :$x^2y'' + (-2xy') + 2y = 6$; $y(0) = 3$ $y'(0) = 1$

$Cx^2+x+3 = 0$ satisfies the initial value problem but $a_n(x) = x^2 = 0$ at $x = 0$

$\Rightarrow$ not a unique solution or it may not be a solution.

- Linear dependence : A set of functions $f_1(x)$, $f_2(x)$,----,$f_n(x)$ is said to be linearly dependent on an interval I if there exist constans c_1, c_2, c_3, ---c_n, not all zeros such that $c_1f_1(x) + c_2f_2(x) + c_3f_3$+---$c_nf_n(x) = 0$ on I.
- The set is linearly independent if it is not linearly dependent or $c_1 = c_2 = c_3 = \text{---} = c_n = 0$.
- A set of functions is linearly dependent on an interval if at least one function can be expressed as a linear combination of the remaining functions.

  ex : $f_1(x) = \sqrt{x} + 5$; $f_2(x) = \sqrt{5} + 5x$; $f_3(x) = x - 1$; $f_4(x) = x^2$ are linearly dependent since $f_2(x) = f_1(x) + 5.f_3(x) + 0.f_4(x) \ \forall$ x in $(0, \infty)$
- **Theorem 4.2 :** For linearly independent funcyions, suppose $f_1(x)$, $f_2(x)$, --- $f_n(x)$ possess at least n-1 derivatives. If the determinant

$$\begin{vmatrix} f_1 & f_2 \dots\dots\dots & f_n \\ f'_1 & f'_2 \dots\dots\dots & f'_n \\ \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_n^{(n-1)} \end{vmatrix} \neq 0$$

for at least one point then the functions are linearly independent.
Then determinant = 0 if functions are linearly dependent.

- A linear n^{th} order differencial equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \text{---} a_1(x)\frac{dy}{dx} + a_o(x)y = 0 \ (3)$$ is said to be homogeneo

$$\text{if } a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \text{---} a_1(x)\frac{dy}{dx} + a_o(x)y = f(x) \ (4)$$ not homogeneous.

- **Theorem 4.3 :** Let y_1, y_2, ---,y_n be solutions of the homogeneous linear nth order D.E. (3) on an interval I. Then $c_1y_1(x) + c_2y_2(x) + c_3y_3$+---$c_ny_n(x)$ is also a solution of the homogeneous D.E.

Corollaries

a) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear D.E. is also a solution.

b) A homogeneous linear D.E. always possesses the trivial solution $y = 0$.

ex : The functions : $y_1= e^x$, $y_2= e^{2x}$ and $y_3= e^{3x}$, all are solutions of the homogeneous equation $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ on $(-\infty, +\infty)$

$\Rightarrow$ by superposition $y = c_1e^x + c_2e^{2x}\ c_3e^{3x}$ is a solution.

ex : $y = x^2$ is a solution of the homogeneous linear D.E. $x^2y''-3xy'+4y = 0$

$\Rightarrow y = cx^2$ is also a solution ($y = e^{2x}$ and $y = 100x^2$)

- $a_n(x)y^{(n)}(x) + a_{(n-1)}(x)y^{(n-1)} +\text{--}+a_o(x)y = g(x)$ (1)
  $a_n(x)y^{(n)}(x) + a_{(n-1)}(x)y^{(n-1)} +\text{--}+a_o(x)y = 0$ (2)
  Let $y_1, y_2, \text{---}, y_n$ be linearly independent solutions of (2)
  Then $y_c = c_1y_1(x) + c_2y_2(x) + c_3y_3 + \text{---} c_ny_n(x)$ is a general solution of (2)
  Let y_p be any particular solution of (1)
  Then $y = y_c + y_p$ is general solution of (1).

- Superposition principle for non homogeneous equations.
  Let $y_{p1}, y_{p2}, \text{----}, y_{pk}$, be the k particular solutions of the linear n^{th} order D.E. (1) on I corresponding to k distinct functions $g_1(x), g_2(x), \text{---}, g_k(x)$ Then $y_p = y_{p1} + y_{p2} + \text{----} + y_{pk}$ is a particular solution of
  $a_n(x)y^{(n)} + a_{(n-1)}(x)y^{(n-1)} +\text{--}+a_o(x)y = g_1(x) + g_2(x) + \text{---} + g_k(x)$
  ex : $y_{p1} = -4x^2$ is particular solution of $y''-3y'+4y = -16x^2+14x - 8$.
  $y_{p2} = e^{2x}$ is particular solution of $y''-3y'+4y = 2e^{2x}$
  $y_{p3} = xe^x$ is particular solution of $y''-3y'+4y = 2xe^x - e^x$
  $\Leftrightarrow y = y_{p1} + y_{p2} + y_{p3} = -4x^2+e^{2x}+xe^x$ is a solution of
  $y''-3y'+4y = -16x^2+24x-8+2^{e2x}+2xe^x-e^x$

- Any set $y_1, y_2, \text{----}, y_n$ of n linear independent solutions of the homogeneous linear n^{th} order equation (1) on I is said to be a fundumental set of solutions.

Section 4.2 Reduction of 2^{nd} order homogeneous D.E.

Let $a_2(x)y''(x) + a_1(x)y' + a_o(x)y = 0$ (1) be divided by $a_2(x)$

$\Rightarrow y''+ p(x)y' + Q(x)y = 0$ (2)

Let $y_1(x)$ be a known solution of (2)

If $y = u(x)y_1(x) \quad \Rightarrow y'' = uy_1'' + 2y_1'u' + y_1u''$

$$y'' + py' + Qy = u\underbrace{(y_1'' + py_1' + Qy_1)}_{0} + y_1u'' + (2y_1' + py_1)u' = 0$$

$\Rightarrow y_1u'' + (2y_1' + py_1)u' = 0 \Rightarrow y_1w + (2y_1' + py_1)w = 0 \; (w = u')$

$$\Rightarrow \frac{dw}{w} + \frac{2y_1'}{y_1}dx + pdx = 0 \qquad \ln|w| + 2\ln|y_1| = -\int pdx + C$$

$$\ln|wy_1^2| = -\int pdx \qquad wy_1^2 = C_1e^{-\int pdx}$$

$$\Rightarrow w = u' = C_1\frac{e^{\int -pdx}}{y_1^2} \qquad u = C_1\int\frac{e^{\int -pdx}}{y_1^2}dx + C_2$$

$$y = u(x)y_1(x) = C_1y_1(x)\int\frac{e^{\int -pdx}}{y_1^2}dx + C_2y_1(x)$$

By choosing $C_2 = 0$ and $C_1 = 1$

$$y_2 = y_1(x)\int\frac{e^{\int -pdx}}{y_1^2}dx$$