MATHEMATICS 202
(1st Semester, 1994-95) *February, 1995*
Final Examination

1) Find a solution of the differential equation :

$$2yy'+x=\frac{e^{(-6y^2-3x^2)/4}}{x^2+1}$$

satisfying y(0) = 0 as follows : Make the substitution $w=e^{y^2}$ to turn the equation into a Bernoulli equation and then obtain the solution .

2) Solve the differential equation :

$$\frac{dy}{dx}=\frac{-x+y\sqrt{x^2+y^2}}{y+x\sqrt{x^2+y^2}}$$

by using an integrating factor of the form $(x^2+y^2)^m$.

3) If $y_1 = u(x)$ and $y_2 = x\,u(x)$ are solutions of the differential equation :

$$y'' + 4xy' - p(x)y = 0 ,$$

and u(0) = 1 , determine u(x) and p(x) explicitly in terms of x .

4) Find the general solution of

a) $y'' - 4y' + 5y = \cos 2x$

b) $x^2y''-2xy'-4y=e^x$

5) Consider the D.E.

$$x(x-1)y'' + (2x-1)y' + (1/4)y = 0$$

a) using power series show that one solution of this equation near 0 is given by :

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{1.3....(2n-1)}{2.4....(2n)} \right)^2 x^n$$

b) Let $y_2 = y_1 \ln x + u(x)$ be another solution of this equation. Find a differential equation satisfied by u . Writing $u(x) = \sum_{n=1}^{\infty} c_n x^n$ determine cn explicitly in terms of n .

6) a) Find the Laplace transform of :

i) $te^{-et} \cos 2t$

ii) $u(t-\pi)\sin t$

iii) $f(t) = \begin{cases} 5, & 3 < t < \infty \\ 1, & 0 \le t \le 3 \end{cases}$

b) Find the inverse Laplace transform of :

$$\frac{s+3}{s^2+8s+17}, \quad \frac{2}{s^3(s+1)}, \quad \frac{s^2+8s-3}{(s^2+2s+1)(s^2+1)}$$

7) Solve by Laplace transform methods

a) $y'' - 3y' - 4y = f(t)$ where $f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

$y(0) = 0, \quad y'(0) = 1$

b) $x' + y = 2\cos t \qquad x(0) = 0$
$x + y' = 0 \qquad y(0) = 1$

8) If J(x) is the Bessel function of order γ , prove that

$$[x^{-\gamma} J_{\gamma}(x)]' = -x^{-\gamma} J_{\gamma+1}(x)$$

Solution
Final 1994-95

1) $2yy'+x=\dfrac{e^{(-6y^2-3x^2)/4}}{x^2+1}=E$

$w=e^{y^2} \quad \Rightarrow \quad w'=2yy'e^{y^2}$

$E=\dfrac{w}{w'}+x=\dfrac{e^{\frac{-3x^2}{4}}w^{-3/2}}{x^2+1}$ Bernoulli

$\dfrac{w'}{w^{-1/2}}+\dfrac{x}{w^{-3/2}}=\dfrac{e^{\frac{-3x^2}{4}}}{x^2+1}$

let u = $w^{3/2} \quad \Rightarrow \quad u'=\dfrac{3}{2}w^{1/2}w'$

$E=u'+\dfrac{3}{2}u=\dfrac{3}{2}\dfrac{e^{\frac{-3x^2}{4}}}{x^2+1}$

linear equation

$e^{\frac{3x^2}{4}}u=\int\dfrac{3}{2}(\dfrac{1}{x^2+1})dx$

$u=(\dfrac{3}{2}arctg.x+c)e^{\frac{-3x^2}{4}}$

2) $\frac{dy}{dx} = \frac{-x + y\sqrt{x^2 + y^2}}{y + x\sqrt{x^2 + y^2}}$ $\qquad$ $(x^2 + y^2)^m = \mathrm{M}$

$M = (x^2 + y^2)^m (y + x\sqrt{x^2 + y^2})$

$\frac{\partial M}{\partial x} = 2mx(x^2 + y^2)^{m-1}(y + x\sqrt{x^2 + y^2}) + (x^2 + y^2)^{m+1/2} + x^2(x^2 + y^2)^{m-1/2}$

$-N = (-x + y\sqrt{x^2 + y^2})(x^2 + y^2)^m$

$\Rightarrow \frac{\partial -N}{\partial y} = 2my(x^2 + y^2)^{m-1}(-x + y\sqrt{x^2 + y^2})^{m-1} + (x^2 + y^2)^{m+1/2} + y^2(x^2 + y^2)^{m-1/2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 2mx(x^2 + y^2)^{m-1}(y + x\sqrt{x^2 + y^2}) + (x^2 + y^2)^{m+1/2} + x^2(x^2 + y^2)^{m-1/2} =$
$2my(x^2 + y^2)^{m-1}(-x + y\sqrt{x^2 + y^2})^{m-1} - (x^2 + y^2)^{m+1/2} - y^2(x^2 + y^2)^{m-1/2} + y^2\sqrt{x^2 + y^2}$

$\Rightarrow \quad 2mxy(x^2 + y^2)^{m-1} + (2m + 1)x^2(x^2 + y^2)^{m-1/2} = 2mxy(x^2 + y^2)^{m-1} + (2m + 1)y^2(x^2 + y^2)^{m-1/2}$
for n $= -1/2$ the equation becomes exact

3) $y_1 = u(x)$ and $y_2 = x\,u(x)$
$y'' + 4xy' - p(x)y = 0$, $\qquad$ $u(0) = 1$

$xu(x) = u(x)\int \frac{e^{-\int 4x dx}}{u(x)^2} \quad \Rightarrow \quad x = \int \frac{e^{-2x^2}}{u(x)^2} dx$

$1 = \frac{e^{-2x^2}}{u(x)^2} \quad \Rightarrow \quad u(x)^2 = e^{-2x^2} \quad \Rightarrow \quad xu(x) = e^{-x}$

$e^{-x} - 4xe^{-x} - p(x)e^{-x} = 0$
$1 - 4x = p(x)$

4)
a) $y'' - 4y' + 5y = \cos 2x$

$m^2 - 4m + 5 = 0$
$y_c = c_1 e^x + c_2 e^{5x}$
$y_p = A\cos 2x + b\sin 2x$
A and B are easily found .

b) $x^2 y'' - 2xy' - 4y = e^x$
caushy euler

5)

a) $x(x-1)y'' + (2x-1)y' + (1/4)y = 0$

Frobenius method

$r(r-1) = 0 \quad => \quad r = 0, \quad r = 1$

$y = \sum c_n x^{n+r}$

$y = \sum_{n=0} c_n x^{n+1} \quad ; \quad y' = \sum_{n=0} (n+1)c_n x^n \; ; \quad y'' = \sum_{n=0} (n+1)nc_n x^{n-1}$

$\sum_{n=0} (n+1)nc_n x^{n+1} + 2\sum_{n=0} (n+1)c_n x^{n+1} + \frac{1}{4}\sum_{n=0} c_n x^{n+1} - \sum_{n=0} (n+1)nc_n x^n - \sum_{n=0} (n+1)nc_n x^n$

$c_0 = 1 \qquad c_0 = c_0$, but $c_0 = 1$ for a particular case of y_1 of the general solution

$$c_n = \left(\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}\right)^2 x^n$$

$$y_1 = 1 + \sum_{n=1}\left(\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}\right)^2 x^n$$

b)

$y_2 = y_1 \ln x + u(x)$

$y_2' = y_1' \ln x + \frac{y_1}{x} + u'(x)$

$y_2'' = \frac{y_1'}{x} + y_1'' \ln x + \frac{y_1'}{x} - \frac{y_1}{x^2} + u''(x)$

$x(x-1)(-y'' \ln x + 2\frac{y_1'}{x} - \frac{y_1}{x^2} + u''(x)) + (2x-1)(y_1' \ln x + \frac{y_1}{x} + u'(x)) + \frac{1}{4} y_1 \ln x + u(x) = 0$

$2y_1'(x-1) + \left(\frac{2x-1}{x}\right) y_1 + x(x-1)u''(x) + (2x-1)u'(x) + \frac{u(x)}{4} = 0$

by deriving we get :

$u'''(x)(x-1) - (2x-2)u'' + (2+1/4)u' = 0$

let $z = u'$ then find $u(x)$

6)

a)

$$\zeta \; te^{-ct} \cos 2t = \left(\frac{s+e}{s^2+4}\right)$$

$$\zeta \; u(t-\pi)\sin(t-\pi) = \frac{e^{-\pi s}}{s^2+1}$$

$\zeta \; (f(t) - 1) = \zeta \; f(t) - \zeta \; (1)$

$\zeta \; (u(t-3)4) = \zeta \; f(t) - 1/s$

$$\zeta \; f(t) = \frac{4e^{-3s}}{s} + \frac{1}{s}$$

b)

$$\zeta\ \frac{s+3}{s^2+8s+17}=e^{-4t}\cos t+3e^{-4t}\sin t$$

$$\zeta\ \frac{2}{s^3(s+1)}=t^2\times e^{-t}\ \text{(Compulsive product.)}$$

$$\frac{s^2+8s-3}{(s+1)^2(s^2+1)}=\frac{As+B}{(s+1)^2}+\frac{Cs+D}{s^2+1}$$

7)

a) y'' - 3y' - 4y = f(t) where $f(t)=\begin{cases}\sin t\ , & 0\le t\le\pi\\ 0\ , & t>\pi\end{cases}$

$y(0)=0\ ,\quad y'(0)=1$

f(t) = sint - u(t-π)sint

ζ(y'' - 3y' - 4y) = ζ f(t)

$$s^2y-1-3sy-4y=\frac{1}{s^2+1}+\frac{e^{-\pi s}}{s^2+1}$$

$$y=\frac{1}{(s^2+1)(s^2-3s-4)}+\frac{e^{-\pi s}}{(s^2+1)(s^2-3s-4)}+\frac{1}{(s^2-3s-4)}$$

$$y=\sin t\times(\frac{1}{5}e^{-t}-\frac{1}{3}e^{-4t})+e^{t}(\frac{1}{5}e^{-t}-\frac{1}{3}e^{-4t})+(\frac{1}{5}e^{-t}-\frac{1}{3}e^{-4t})$$

b) x' + y = 2 cost x(0) = 0

x + y' = 0 y(0) = 1

$$sx-y=\frac{2s}{s^2+1}$$

$$x-sy+1=0$$

8)

$$(x^{-\nu}J_\nu x)' = -x^{-\nu}J_{\nu+1}$$

$$J_\nu x = \sum \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \times \left(\frac{x}{2}\right)^{2n+\nu}$$

$$(x^{-\nu}J_\nu x)' = -\nu x^{-\nu-1}J_\nu x + J_\nu' x^{-\nu}$$

$$= -\nu x^{-(\nu+1)}J_\nu x + x^{-\nu}\sum \frac{(-1)^n}{n!\Gamma(1+\nu+n)}\frac{1}{2} \times \left(\frac{x}{2}\right)^{2n+\nu-1}$$

change n into n + 1 we get :

$$(x^{-\nu}J_\nu x)' = -x^{-\nu}J_{\nu+1}(x)$$