MATHEMATICS 202
(2nd Semester, 1992-93)
Quiz II May 13, 1993

1) Apply the Frobenius method to solve the equation :

$$x^2y''-2xy'+(2-x^2)y=0$$

and identify the series obtained as expansions of known functions .

2) Solve the initial value problem :

$$x' = -2x + y$$
$$y' = -4x + 3y + 10\cos t$$

with $x(0) = 1$; $y(0) = -6$

3) Use the Laplace transform method to solve :

$$y''(t) - 4y'(t) + 4y(t) = 4\cos 2t$$

with $y(0) = 2$; $y'(0) = 5$.

4) Consider the differential equation :

$$y''+4x^8y=0 \qquad \ldots \quad (A)$$

Use the substitution $y(x)=x^a u(x)$ where u is a function of x, and find the differential equation satisfied by u . Then determine a so that the equation satisfied by u has the form :

$$x^2u''+xu'+(4x^{10}-1/4)u=0 \qquad \ldots \quad (B)$$

Make the substitution $z=bx^5$ and determine b so that equation (B) is turned into a Bessel equation (C) .

c) Deduce from parts (a) and (b) the general solution of equation (A) .

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Solution
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1) $x^2y''-2xy'+(2-x^2)y=0$

xp(x) = -2 ; $x^2q(x) = 2 - x^2$ at v = 2

0 is a singular point .

$r(r-1) + p_0r + q_0 = 0$

$r^2 - 3r + 2 = 0$ => r = 1 & r = 2

$y_1=\sum_{n=0}^{\infty}c_n x^{n+r}$; $y'=\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$

$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$

$x^r\left(\sum_{o}^{\infty}(n+r)(n+r-1)c_n x^n-2\sum_{n=0}^{\infty}(n+r)c_n x^n+2\sum_{n=0}^{\infty}c_n x^n-\sum_{n=0}^{\infty}c_n x^{n+2}\right)$

for the first three n = k & n + 2 = k => n = k - 2

$x^r\left(\sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^k-2\sum_{k=0}^{\infty}(k+r)c_k x^k+2\sum_{k=0}^{\infty}c_k x^k-\sum_{k=2}^{\infty}c_{k-2}x^k\right)$

r(r-1) - 2r + 2 = 0 ; r = 1, & r = 2

r = 2 => $2c_1 = 0$ => c1 = 0

r = 1 => 0 c1 = 0 undetermined

(k = 2)

$[(k+r)(k+r-1) - 2(k+r) + 2]c_k + c_{k-2} = 0$

$c_k=\dfrac{c_{k-2}}{(k+r)(k+r-3)+2}$; for (r = 2)

$c_k=\dfrac{c_{k-2}}{(k+2)(k-1)+2}$

$c_2=c_0/6$; $c_3=0$; $c_4=c_0/120$; $c_5=0$

$c_{2n}=\dfrac{c_0}{(2n+1)!}$

$y_1=x\sin x$

$y_2=y_1\int\dfrac{e^{-\int p(x)dx}}{y_1^{\,2}}dx=-x\sin x\cot gx$

2) x' = -2x + y

y' = -4x + 3y + 10 cost

y" = -4x' + 3y' -10sint

y" = 8x - 4y - 12x +9y - 10sint

$y'' = y' - 7y - 10(\cos t + \sin t)$

$y'' - y' + 7y = 0$

$m^2 - m + 7 = 0 \qquad \Delta = -6$

$m_1 = \frac{1}{2} - \frac{\sqrt{6}}{2}i \; ; \qquad m_2 = \frac{1}{2} + \frac{\sqrt{6}}{2}i$

$y_1 = e^{t/2}(c_1 \cos\frac{\sqrt{6}}{2}t + c_2 \cos\frac{\sqrt{6}}{2}t)$

$y_p = A\cos t + B\sin t$

solving and comparing with the original eqn. we get :

A = 5 / 37 ; B = - 70 / 37

General Solution $= e^{t/2}(c_1 \cos\frac{\sqrt{6}}{2}t + c_2 \cos\frac{\sqrt{6}}{2}t) + A\cos t + B\sin t$

3) $y''(t) - 4y'(t) + 4y(t) = 4\cos 2t$

$y(0) = 2 ; \quad y'(0) = 5 .$

$\zeta(y'' - 4y' + 4y) = \zeta(4\cos 2t)$

$$s^2 y - 2s - s - 4sy + 8 + 4y = \frac{4s}{s^2 + 4}$$

$$y = \zeta^{-1}\left(\frac{4s}{(s^2 + 4)(s^2 - 4s + 8)} + \frac{2s - 3}{s^2 - 4s + 8}\right)$$

$$y = \cos 2t \underset{\Downarrow}{\times} 2\sin 2te^{2t} + 2e^{2t}\cos 2t - \frac{3}{2}e^{2t}\sin 2t$$

Compulsive product.

4)

a) $y'' + 4x^8 y = 0 \qquad \ldots \quad (A)$

$y(x) = x^a u(x)$

$y' = ax^{a-1}u(x) + u'(x)x^a$

$y'' = a(a-1)x^{a-2}u(x) + +u'(x)(ax^{a-1}) + u''(x)a + au'(x)x^{a-1}$

$\Rightarrow \quad x^a u'' + 2au'x^{a-1} + a(a-1)x^{a-2}u + 4x^{a+8}u = 0$

let a = 1/2

$x^{1/2}u'' + u'x^{-1/2} + u(4x^{8+1/2} - (1/4)x^{-3/2})$

multiply by $x^{3/2}$

$x^2 u'' + xu' + u(4x^{10} - 1/4)$

b)

$$z = bx^5 \qquad \partial z = 5bx^4 \partial x$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} = \frac{u'}{5bx^4} \Rightarrow u' = 5bx^4 \frac{\partial u}{\partial z}$$

$$\frac{\partial^2 u}{\partial^2 z} = \frac{(\partial u'/5bx^4)\partial x}{\partial x \ \partial z} = \frac{u''(5bx^4) - 20bx^3u'}{(5bx^4)^2} \times \frac{1}{5bx^4}$$

$$\frac{\partial^2 u}{\partial^2 z}(5bx^4)^2 \times x^2 + (20bx^3u') \times x^2 + 5bx^4 \frac{u(x)}{\partial z} + u(x)(4x^{10} - 1/4)$$

$$\frac{\partial^2 u}{\partial^2 z}(25b^2x^{10} + 20bx^5 + 5bx^5)\frac{\partial u}{\partial z} + u(x)(4x^{10} - 1/4)$$

$$let \ b = 5$$

$$\Rightarrow z^2u'' + zu' + u(\frac{4}{25}z^2 - 1/4)$$

c) Bessel equation

$$u = c_1 J_{1/2}(\frac{2}{5}z) + c_2 J_{-1/2}(\frac{2}{5}z)$$