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Sec. 1 & 2

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Math 202

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Time : 1 hour

(1-st Semester, 1998-1999)

QUIZ II

(closed book)

④ I. Solve the BVP

$$y'' + y = x^2 + 1 ; \quad y(0) = 5, \quad y(1) = 0.$$

④ II. Solve the IVP

$$x^2 y'' + x y' - y = 4x ; \quad y(1) = y'(1) = 0.$$

④ III. Find the general power series solution of the following DE about the ordinary point $x = 0$.

$$(x^2 - 1) y'' + 4x y' + 2y = 0.$$

④ IV. Apply the method of Frobenius to find one of the solutions to the DE

$$(x - 1) y'' + y = 0,$$

in the neighborhood of its $x = 1$ singularity.

(5)

$$\text{I. } y'' + y = x^2 + 1 \quad ; \quad y(0) = 5, \quad y(1) = 0$$

$$\text{ch. Eqn. } m^2 + 1 = 0 \Rightarrow m_{1,2} = \pm i$$

$$y_c = c_1 e^{ix} + c_2 \sin x \quad (+)$$

$$f(x) = x^2 + 1 = P_2(x)$$

$\omega = 0$ & $\beta = 0$ & $x=0$ is not root C.E. $\therefore x^2 \neq 1$ no secular term.

$$y_p(x) = \hat{P}_2(x) = Ax^2 + Bx + C \quad (1)$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$2A + Ax^2 + Bx + C = x^2 + 1$$

$$Ax^2 + Bx + (2A+C) \equiv x^2 + 1$$

$$A = 1, \quad B = 0 \quad \& \quad 2A+C = 1$$

$$y_p = x^2 - 1 \quad (1)$$

$$y = y + y_p = c_1 e^{ix} + c_2 \sin x + x^2 - 1 \quad (1)$$

$$y(0) = c_1 + c_2 \cdot 0 - 1 = 5 \Rightarrow c_1 = 6 \quad (+)$$

$$\rightarrow y = 6 \cos x + c_2 \sin x + x^2 - 1$$

$$y(1) = 6 \cos 1 + c_2 \sin 1 = 0$$

$$\therefore c_2 = -6 \frac{\cos 1}{\sin 1} = -6 \cot 1 \quad (+)$$

$$\rightarrow y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1 \quad (+)$$

D II

$$x^2 y'' + xy' - y = 4x \quad ; \quad y(1) = y'(1) = 0$$

$$y = x^m \Rightarrow x^m(m^2 - 1) = 0$$

$$y_c(x) = c_1 \underbrace{x}_y + c_2 \underbrace{x^{-1}}_{y_2} \quad (1)$$

$$y_p = u_1 x + u_2 x^{-1} \quad (+)$$

$$\mathcal{L}[y] = y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \frac{4}{x} = f(x)$$

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \Rightarrow \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{4}{x} \end{pmatrix} \quad (1)$$

$$w = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1} = -\frac{2}{x}$$

$$w_1 = \begin{vmatrix} 0 & \frac{1}{x} \\ \frac{4}{x} & -\frac{1}{x^2} \end{vmatrix} = -\frac{4}{x^2}, \quad u_1 = \frac{w_1}{w} = \frac{-\frac{4}{x^2}}{-\frac{2}{x}} = \frac{2}{x}$$

$$u_1 = 2 \int \frac{dx}{x} = 2 \ln x$$

$$w_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{4}{x} \end{vmatrix} = 4, \quad u_2 = \frac{w_2}{w} = \frac{4}{-\frac{2}{x}} = -2x$$

$$u_2 = -2 \int x dx = -x^2$$

$$y_p = 2x \ln x - x \quad (1+)$$

$$y = y_c + y_p = c_1 x + c_2 x^{-1} + 2x \ln x - x$$

$$y = c_1 x + c_2 x^{-1} + 2x \ln x \quad (1)$$

$$\boxed{y(1) = c_1 + c_2 = 0} \quad (+)$$

$$y' = c_1 - c_2 x^{-2} + 2 \ln x + 2$$

$$\boxed{y'(1) = c_1 - c_2 + 2 = 0}$$

$$2c_1 + 2 = 0 \Rightarrow \boxed{c_1 = -1} \quad (+)$$

$$c_2 = -c_1 = 1$$

$$(1) \quad y = -x + x^{-1} + 2x \ln x$$

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III

$$(x^2 - 1)y'' + 4xy' + 2y = 0$$

$$y = \sum_{n=0}^{\infty} C_n x^n \quad (1) \quad , \quad y' = \sum_{n=0}^{\infty} n C_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2}$$

$$(x^2 - 1) \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} + 4x \sum_{n=0}^{\infty} n C_n x^{n-1} + 2 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) C_n x^n - \underbrace{\sum_{n=0}^{\infty} n(n-1) C_n x^{n-2}}_{\begin{matrix} n=k-2 \\ n=k+2 \\ n-1=k+1 \end{matrix}} + 4 \sum_{n=0}^{\infty} n C_n x^n + 2 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) C_n x^n - \sum_{n=0}^{\infty} (n+k)(n+k+1) C_{n+k+2} x^n + 4 \sum_{n=0}^{\infty} n C_n x^n + 2 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n \left\{ n(n-1) C_n - (n+k)(n+k+1) C_{n+k+2} + 4n C_n + 2 C_n \right\} = 0$$

$$\sum_{n=0}^{\infty} x^n \left\{ C_n \underbrace{[n(n-1) + 4n + 2]}_{n^2 - n + 4n + 2} - (n+k)(n+k+1) C_{n+k+2} \right\} = 0$$

$$n^2 - n + 4n + 2 = n^2 + 3n + 2 = (n+2)(n+1)$$

$$C_k (k+1)(k+2) - (k+2)(k+1) C_{k+2} = 0$$

$$(2) \quad \boxed{C_{k+2} = C_k} \quad \text{recurrence relation}$$

$$C_0 \neq 0$$

$$n=0 : C_2 = C_0$$

$$n=1 : C_3 = C_1$$

$$C_1 \neq 0$$

$$n=2 : C_4 = C_2 = C_0$$

$$n=3 : C_5 = C_3$$

$$n=4 : C_6 = C_4 = C_2 = C_0$$

$$n=5 : C_7 = C_5 = C_3$$

$$\therefore C_{2m} = C_0 ; m = 0, 1, 2, 3, \dots \quad (1)$$

$$C_{2m+1} = C_1 ; m = 0, 1, 2, 3, \dots \quad (1)$$

$$y = C_0 \sum_{n=0}^{\infty} x^{2n} + C_1 \sum_{n=0}^{\infty} x^{2n+1} \quad (1)$$

$$y = A \frac{1}{1-x^2} + B \frac{x}{1-x^2} ; \quad x^2 < 1 \quad (1)$$

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IV

$$(x-1) y'' + y = 0 \quad x=1 \text{ is a potential singularity}$$

$$z = x-1, dz = dx, dz^2 = dx^2$$

$$\ddot{y} = \ddot{y} \quad \text{where } \ddot{y} = \frac{d^2y}{dz^2}$$

$$z\ddot{y} + y = 0 \quad (1) \rightarrow \ddot{y} + \frac{1}{z} y = 0$$

$$P(z) = z P(z) = 0 \equiv \sum_{n=0}^{\infty} p_n z^n \Rightarrow p_0 = 0; p_n = 0, \forall n \geq 1$$

$$Q(z) = z^2 Q(z) = z \equiv \sum_{n=0}^{\infty} q_n z^n \Rightarrow q_0 = 0, q_1 = 1; q_n = 0, \forall n \geq 2$$

$$y(z) = \sum_{n=0}^{\infty} c_n(r) z^{n+r}$$

$$F(r) = r^2 + (p_{r-1})r + q_0 = r^2 - r = r(r-1) = 0$$

$$r_1 = 1, r_2 = 0 \quad (1) \quad r_1 - r_2 = N \in \mathbb{N} \quad \Rightarrow \text{b}_0 \text{ is a critical coefficient.}$$

$$b_n = -\frac{1}{n(n-1)} \sum_{s=0}^{n-1} [(s+r_2) p_{n-s} + q_{n-s}] b_s = -\frac{1}{n(n-1)} \sum_{s=0}^{n-1} q_{n-s} b_s$$

$$b_0 \neq 0 \quad b_1 = -\frac{1}{1(1-1)} q_1 b_0 = -\frac{1}{0} \rightarrow \infty \quad (1)$$

$$(1) \quad C_n = -\frac{1}{n(n+1)} \sum_{s=0}^{n-1} q_{n-s} c_s \quad c_0 \neq 0 \quad (1)$$

$$c_1 = -\frac{1}{1 \cdot 2} q_1 c_0 = -\frac{c_0}{1 \cdot 2}$$

$$c_2 = -\frac{1}{2 \cdot 3} [q_2 c_0 + q_1 c_1] = -\frac{c_1}{2 \cdot 3} = \frac{c_0}{(1 \cdot 2)(2 \cdot 3)}$$

$$c_3 = -\frac{1}{3 \cdot 4} [q_3 c_0 + q_2 c_1 + q_1 c_2] = -\frac{c_2}{3 \cdot 4} = -\frac{c_0}{(1 \cdot 2 \cdot 3)(2 \cdot 3 \cdot 4)}$$

$$c_4 = -\frac{1}{4 \cdot 5} [q_4 c_0 + q_3 c_1 + q_2 c_2 + q_1 c_3] = -\frac{c_3}{4 \cdot 5} = +\frac{c_0}{4! s!}$$

$$c_5 = -\frac{c_0}{5! s!} \quad \text{suggestion} \Rightarrow \text{verified}$$

$$C_n = \frac{(-1)^n c_0}{n! (n+1)!} \quad (1)$$

$$y_1(z) = z \sum_{n=0}^{\infty} c_n z^n = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n! (n+1)!} \quad (1)$$

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n! (n+1)!} \quad (1)$$