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Sec. 1 & 2

Time : 1 hour

FACULTY OF ARTS & SCIENCES , A. U. B.

Math 202

(1-st Semester , 1997-1998)

QUIZ II

19 Jan.,1998

(closed book)

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(5) I. Solve the IVP

$$y''' + 7y'' + 19y' + 13y = 0 \quad ; \quad y(0) = 0 \,,\; y'(0) = 2 \,,\; y''(0) = -12 \,.$$

(5) II. Find the general solution of the DE

$$y'' + y = \csc x + x^2$$

(5) III. Apply the method of Frobenius to find the general solution of the DE

$$x(1-x)\,y'' - 3y' + 2y = 0 \,,$$

in the neighborhood of its $x = 1$ singularity .

(5) IV. Use the substitution $y = \sqrt{x}\;u(x)$ to reduce the DE

$$y'' + [c^2 - k(k-1)/x^2]\,y = 0 \;;\; k \in \mathbb{N} \,,\, c > 0 \,,$$

to an equivalent Bessel's DE , then express its solution in terms of Bessel functions .

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i) I

IVP: $y''' + 7y'' + 19y' + 13y = 0$; $y(0)=0, y'(0)=2, y''(0)=-12$

A.E. $m^3 + 7m^2 + 19m + 13 = 0$ (+)

$$\begin{array}{r|l} & m^2 + 6m + 13 \\ \hline m+1 & m^3 + 7m^2 + 19m + 13 \\ & -\cancel{m^3} \mp m^2 \\ \hline & 6\cancel{m^2} + 19m + 13 \\ & -6\cancel{m^2} \mp 6m \\ \hline & 13\cancel{m} + 1\cancel{3} \\ & 13\cancel{m} + 1\cancel{3} \\ \hline \end{array}$$

$(m+1)(m^2 + 6m + 13) = 0$ (1)

$m_1 = -1$

$m_{2,3} = -3 \pm \sqrt{9-13} = -3 \pm 2i$

$y = c_1 e^{-x} + e^{-3x}(c_2 \cos 2x + c_3 \sin 2x)$ (1)

$y(0) = c_1 + c_2 = 0$ (+)

$y' = -c_1 e^{-x} - 3e^{-3x}(c_2 \cos 2x + c_3 \sin 2x) + e^{-3x}(-2c_2 \sin 2x + 2c_3 \cos 2x)$

$y'(0) = -c_1 - 3c_2 + 2c_3 = 2$ (+)

$y'' = c_1 e^{-x} + 9e^{-3x}(c_2 \cos 2x + c_3 \sin 2x) - 3e^{-3x}(-2c_2 \sin 2x + 2c_3 \cos 2x) + e^{-3x}(-4c_2 \cos 2x - 4c_3 \sin 2x)$

$-3e^{-3x}(-2c_2 \sin 2x + 2c_3 \cos 2x)$

$y''(0) = c_1 + 9c_2 - 6c_3 - 6c_3 - 4c_2 = -12$ (+)

$c_1 + 5c_2 - 12c_3 = -12$

$\therefore c_3 = 1$, $c_1 = -c_2 = 0$. (+)

IVP: $y = e^{-3x} \sin 2x$ (+)

II

$$y'' + y = \csc x + x^2$$

C.E. $m^2 + 1 = 0 \quad \therefore m_{1,2} = \pm i$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2} \quad (1)$$

$$L[y] = f(x) = \csc x + x^2 = f_1(x) + f_2(x)$$

$$\left.\begin{aligned} L[y_{p1}] &= f_1(x) = \csc x \\ L[y_{p2}] &= f_2(x) = x^2 \end{aligned}\right\} \quad (1)$$

For $L[y_{p1}] = f_1(x)$ use the method of Lagrange $\Rightarrow$ $y_{p1} = u_1 y_1 + u_2 y_2$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f_1(x) \end{pmatrix} ; \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \csc x \end{pmatrix}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \quad (+)$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix} = -1 \ (+), \quad u_1' = \frac{W_1}{W} = -1, \quad u_1 = \int u_1'(x)\,dx = -x \quad (+)$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin x} \end{vmatrix} = \frac{\cos x}{\sin x}, \quad u_2' = \frac{W_2}{W} = \frac{\cos x}{\sin x}, \quad u_2 = \int u_2'(x)\,dx = \int \frac{d(\sin x)}{\sin x}$$

$$\therefore u_2 = \ln|\sin x| \quad (+)$$

$$y_{p1} = -x\cos x + \sin x \ln|\sin x|$$

For $L[y_{p2}] = f_2(x)$ use the method of undetermined cofficients

$f_2(x) = x^2 = P_2(x) \ \therefore \Leftarrow \quad \lambda = 0$ is not root C.E.

$$\left.\begin{aligned} \therefore y_{p2}(x) &= \tilde{P}_2(x) = Ax^2 + Bx + C \\ y_{p2}' &= 2Ax + B \\ y_{p2}'' &= 2A \end{aligned}\right\} \quad \begin{aligned} &2A + Ax^2 + Bx + C = x^2 \\ &\therefore A = 1 \\ &\quad B = 0 \\ &C + 2A = 0 \Rightarrow C = -2A = -2 \end{aligned}$$

$$\therefore \ y_{p2} = x^2 - 2 \quad (1)$$

$$y_p = y_{p1} + y_{p2} = -x\cos x + \sin x \ln|\sin x| + x^2 - 2 \quad (1)$$

$$y = y_c + y_p = c_1\cos x + c_2\sin x - x\cos x + \sin x \ln|\sin x| + x^2 - 2$$

III $\quad x(1-x)y'' - 3y' + 2y = 0 \quad \Rightarrow \quad y'' - \frac{3}{x(1-x)}y' + \frac{2}{x(1-x)}y = 0$.

$(1-x) = t \quad , \quad x = 1-t \quad , \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{dy}{dt} \quad ; \quad y' = -\dot{y}$

$y'' = \ddot{y} \quad \therefore \quad \ddot{y} + \frac{3}{t(1-t)}\dot{y} + \frac{2}{t(1-t)}y = 0$. 1

$t=0$

$) = tP = \frac{3}{1-t} \equiv 3\sum_{n=0}^{\infty} p_n t^n = 3 + 3t + 3t^2 + \dots \quad : \quad p_0 = 3 \; ; \; p_n = 3, \; \forall n \geq 1$ (+)

$) = t^2 Q = \frac{2t}{1-t} = 3t\sum_{n=0}^{\infty} q_n t^n = 2t + 2t^2 + 2t^3 + \dots \quad : \quad q_0 = 0 \; ; \; q_n = 2, \; \forall n \geq 1$ (+)

$\therefore y = t^r \sum_{n=0}^{\infty} c_n(r) t^n$

$) = r^2 + (p_0 - 1)r + q_0 = r^2 + 2r = r(r+2) = 0$

$= 0 \, , \, r_2 = -2 \quad : \quad r_1 - r_2 = 2 = N \in \mathbb{N}$ 1

$_n = \frac{-1}{n(n-2)} \sum_{s=0}^{n-1} [(s-2)\underset{3}{\cancel{p_{n-s}}} + \underset{2}{\cancel{q_{n-s}}}] b_s = -\frac{1}{n(n-2)} \sum_{s=0}^{n-1} (3s-4) b_s$ 1

$_0 \neq 0$

$_1 = \frac{-1}{1(1-2)} \; -4b_0 = -4b_0$

$b_2 = \frac{-1}{2(2-2)}[-4b_0 - b_1] = \frac{0}{0} \quad \Rightarrow \quad b_2 \neq 0$. 1

$y_2(t) = y(t) = t^{-2}\sum_{n=0}^{\infty} b_n t^n$

$b_3 = -\frac{1}{3(3-2)}[-4b_0 - b_1 + 2b_2] = -\frac{2}{3}b_2$

$b_4 = -\frac{1}{4(4-2)}[-4b_0 - b_1 + 2b_2 + 5b_3] = \frac{1}{6} b_2$

$b_5 = -\frac{1}{5(5-2)}[-4b_0 - b_1 + 2b_2 + 5b_3 + 8\cdot b_4] = 0$ 1

$b_6 = b_7 = \dots = 0$

$y(t) = t^{-2}[b_0 - 4b_0 t] + t^{-2}[b_2 t^2 - \frac{2}{3}b_2 t^3 + \frac{1}{6}b_2 t^4]$ 1

$y(t) = b_0[t^{-2} - 4t^{-1}] + b_2[1 - \frac{2}{3}t + \frac{1}{6}t^2]$.

$y(t) = b_0[t^{-2} - 4t^{-1}] + b_2[1 - \frac{2}{3}(1-x) + \frac{1}{6}(1-x)^2]$. 1

$\downarrow$

$y(x) = A\left[\frac{1}{(1-x)^2} - \frac{4}{(1-x)}\right] + B\left[1 - \frac{2}{3}(1-x) + \frac{1}{6}\right.$

IV $\quad y = x^{\frac{1}{2}} u \quad$ in $\quad y'' + [c^2 - k(k-1)/x^2] y = 0 \;, \; k \in$

$y' = \frac{1}{2} x^{-\frac{1}{2}} u + x^{\frac{1}{2}} u'$ 1

$y'' = -\frac{1}{4} x^{-\frac{3}{2}} u + \frac{1}{2} x^{-\frac{1}{2}} u' + \frac{1}{2} x^{-\frac{1}{2}} u' + x^{\frac{1}{2}} u''$ +

$y'' = x^{\frac{1}{2}} u'' + x^{-\frac{1}{2}} u' - \frac{1}{4} x^{-\frac{3}{2}} u$

$^{\frac{1}{2}} u'' + x^{-\frac{1}{2}} u' - \frac{1}{4} x^{-\frac{3}{2}} u + [c^2 - \frac{k(k-1)}{x^2}] x^{\frac{1}{2}} u = 0$

$y \cdot x^{3/2}$

$^2 u'' + x u' - \frac{1}{4} u + x^2 [c^2 - \frac{k(k-1)}{x^2}] u = 0$

$^2 u'' + x u' + [c^2 x^2 - k(k-1) - \frac{1}{4}] u = 0$

$^2 u'' + x u' + [c^2 x^2 - (k^2 - k + \frac{1}{4})]$

$^2 - k + \frac{1}{4} = (k - \frac{1}{2})^2$ 1

$x^2 u'' + x u' + [x^2 c^2 - (k - \frac{1}{2})^2] u = 0$

$xc = z \qquad \therefore \quad u' = \frac{du}{dx} = c \frac{du}{dz} = c \dot{u}$ 1

$x = \frac{z}{c} \qquad u'' = c^2 \frac{d^2 u}{dz^2}$

$x^2 = \frac{z^2}{c^2}$ +

$z^2 \ddot{u} + z \dot{u} + [z^2 - (k - \frac{1}{2})^2] u = 0$

$u(z) = A\, J_{k-\frac{1}{2}}(z) + B\, J_{-k+\frac{1}{2}}(z)$

$\Downarrow$

$u(x) = A\, J_{k-\frac{1}{2}}(cx) + B\, J_{-k+\frac{1}{2}}(cx)$ 1

$\frac{y(x)}{x^{\frac{1}{2}}}$

$\therefore \; y(x) = A\, x^{\frac{1}{2}} J_{k-\frac{1}{2}}(cx) + B\, x^{\frac{1}{2}} J_{-k+\frac{1}{2}}(cx)$ 1