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Sec. 7 & 8

FACULTY OF ARTS & SCIENCES , A. U. B.
Math 202
(2-nd Semester, 1997-1998)
QUIZ I

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Time : 1 hour

(closed book)

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❹ I. Solve the DE
$\cos x \cos y \, dx - \sin x \sin y \, dy = 0$.

❹ II. Show that the IVP
$x y' + y = 2x$; $y(0) = 0$,
has exactly one solution but the IVP
$x y' + y = 2x$; $y(0) = y_0 \neq 0$,
has no solution at all. Why doesn't this contradict the related Existence & Uniqueness theorem ?.

❹ III. Find the general solution of the DE
$y' + x^{-1} y = y^{-4}$.

❽ IV. Given that $y_1 = x$ is a solution to both of the following DE's
a) $y' = x^3 (y - x)^2 + x^{-1} y$,
b) $x^2 y'' - 2x y' + 2y = 0$.
Find the general solution to each of these DE's.

❹ V. Solve the IVP
$y y'' = (y')^2 - y'$; $y(0) = 1$, $y'(0) = 2$.

❸ VI. Find the solution of the IVP
$y'' + y' - 6y = 0$; $y(-1) = 1$, $y'(-1) = -1$.

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(4) I. $\cos x \cos y \, dx - \sin x \sin y \, dy = 0$

$$\frac{\cos x}{\sin x} dx = \frac{\sin y}{\cos y} dy \quad (1)$$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\sin y}{\cos y} dy \Rightarrow \int \frac{d(\sin x)}{\sin x} = -\int \frac{d(\cos y)}{\cos y}$$

$$\therefore \ln|\sin x| = -\ln|\cos y| + \ln c \quad (1)$$

$\ln(\sin x) = \ln\left(\frac{c}{\cos y}\right)$ (+); under the assumption that a proper choice of c will take care of details involved in multi-value funct (+)

$$\therefore \sin x = \frac{c}{\cos y} \Rightarrow \cos y = \frac{c}{\sin x}$$

$$y = \cos^{-1}\left(\frac{c}{\sin x}\right) \underset{(1)}{=} \cos^{-1}(c \csc x)$$

Alternatively: $\cos x \cos y \, dx - \sin x \sin y \, dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\cos x \sin y \\ \frac{\partial N}{\partial x} &= -\cos x \sin y \end{aligned}\right\} D = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \; : \text{ DE is exact. } (1)$$

solution: $g(x,y) = \int M \partial x + \int \left[N - \frac{\partial}{\partial y}\int M \partial x\right] dy = c$ (+)

$$\int M \partial x = \int \cos x \cos y \, \partial x = \cos y \int \cos x \, dx = \cos y \sin x \quad (+)$$

$$I = \frac{\partial}{\partial y}\int M \partial x = -\sin y \sin x \quad (+)$$

$$\therefore N - I = 0$$

$$g(x,y) = \cos y \sin x = c$$

$$\cos y = \frac{c}{\sin x} \quad (+)$$

$$y = \cos^{-1}\left(\frac{c}{\sin x}\right) \underset{(1)}{=} \cos^{-1}(c \csc x)$$

II. $xy' + y = 2x$; $y(0) = 0$ is a standard IVP. (+)

$y' + \frac{1}{x} y = 2$; $y(0) = 0$, $\mu = e^{\int \frac{dx}{x}} = x$

general solution: $y = x^{-1}[\int 2x\,dx + c] = x^{-1}[x^2 + c] = x + cx^{-1}$

$y(0) = 0 + c \cdot \infty = 0$ if $c = 0$

$\therefore \boxed{y = x}$ is the unique solution to the IVP. (1)

Now for $xy' + y = 2x$; $y(0) = y_0 \neq 0$ is also a standard IVP

$y(0) = 0 + c \cdot \infty \Rightarrow \infty \neq y_0$, $\forall c \neq 0$

$y(0) = 0 \neq y_0$, $c = 0$

$\therefore$ This IVP has no solution at all (1)

This result does not violate the Exis. & Uniq. Th.

because $a_1(x) = x$: vanishes at $x = 0$. (1)

As for Picard's Th. it is irrelevant because the sufficient conditions on the continuity of

$f(x,y) = \frac{2x - y}{x} = 2 - \frac{y}{x}$

and

$\frac{\partial f}{\partial y} = -\frac{1}{x}$

are not satisfied since at $x = 0$

$\lim_{x \to 0} \frac{\partial f}{\partial y} \to \infty$. (+)

(4) III.

$$y' + x^{-1} y = y^{-4}$$

This is a Bernoulli DE with $n = -4$ (+)

$$\omega = y^{1-n} = y^5 \quad (1)$$

$\Downarrow$

$$\frac{d\omega}{dx} + 5(x^{-1})\,\omega = 5$$

$$\frac{d\omega}{dx} + \frac{5}{x}\,\omega = 5 \quad (1)$$

$$\mu = e^{5\int \frac{dx}{x}} = e^{5\ln x} = x^5$$

$$\omega = x^{-5}\left[\int 5x^5\,dx + c\right]$$

$$\omega = x^{-5}\left[\frac{5}{6}x + c\right] = \frac{5}{6}x + \frac{c}{x^5}$$

$$y^5 = \frac{5}{6}x + \frac{c}{x^5} \quad (1)$$

$$y = \sqrt[5]{\frac{5}{6}x + \frac{c}{x^5}} \quad (+)$$

6 IV

3+ a) $y' = x^3 (y-x)^2 + x^{-1} y \quad ; \quad y_1 = x$

$y' = x^3(y^2 - 2xy + x^2) + x^{-1} y = x^3 y^2 - 2x^4 y + x^5 + \frac{1}{x} y$

$y' = \underbrace{x^5}_{P} + \underbrace{\left(\frac{1}{x} - 2x^4\right)}_{Q} y + \underbrace{x^3}_{R} y^2$ + : Riccati DE

$y = y_1 + u \quad , \quad w = u^{-1}$ +

$w' + (Q + 2y_1 R) w = -R$ +

$w' + \frac{1}{x} w = -x^3$ +

$\mu = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$w = x^{-1} \left[\int x(-x^3) dx + c \right] = x^{-1} \left[-\int x^4 dx + c \right] = x^{-1} \left[-\frac{x^5}{5} + c \right]$

$w = \frac{c}{x} - \frac{x^4}{5} = u^{-1}$ +

$\therefore u = \dfrac{1}{\frac{c}{x} - \frac{x^4}{5}} = \dfrac{5x}{5c - x^5} = \dfrac{5x}{c - x^5}$

$y = y_1 + u = x + \dfrac{1}{\frac{c}{x} - \frac{x^4}{5}} = x + \dfrac{5x}{c - x^5}$ 1

2+ b) $x^2 y'' - 2xy' + 2y = 0 \quad ; \quad y_1 = x$

$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0$ +

$y_2 = y_1 \int \dfrac{e^{-\int P(x) dx}}{y_1^2(x)} dx = x \int \dfrac{e^{2\int \frac{dx}{x}}}{x^2} dx$ +

$y_2 = x \int \frac{x^2}{x^2} dx = x \int dx = x^2$ +

$\therefore y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^2$ 1

(3)

VI $\quad y'' + y' - 6y = 0 \quad ; \quad y(-1) = 1 \quad , \quad y'(-1) = -1$

C.E.: $\quad m^2 + m - 6 = 0$ (+)

$\therefore \quad (m+3)(m-2) = 0 \quad \Rightarrow \quad m_1 = -3 \, , \quad m_2 = 2$

General solution: $\quad y = c_1 e^{-3x} + c_2 e^{2x}$ (1)

$y(-1) = c_1 e^{3} + c_2 e^{-2} = 1$

$y' = -3c_1 e^{-3x} + 2c_2 e^{2x}$

$$\begin{array}{r} y'(-1) = -\cancel{3c_1 e^{3}} + 2c_2 e^{-2} = -1 \\ \cancel{3c_1 e^{3}} + 3c_2 e^{-2} = 3 \\ \hline 5c_2 e^{-2} = 2 \end{array}$$

$\therefore \quad c_2 = \frac{2}{5} e^{2}$

$c_1 e^{3} + \frac{2}{5} e^{2} e^{-2} = 1 \quad \Rightarrow \quad c_1 e^{3} = \frac{3}{5}$ (+)

$\therefore \quad c_1 = \frac{3}{5} e^{-3}$

IVP solution:

$$y = \frac{3}{5} e^{-3} e^{-3x} + \frac{2}{5} e^{2} e^{2x}$$

$$y = \frac{3}{5} e^{-3x-3} + \frac{2}{5} e^{2x+2}$$ (1)

4 V $\quad yy'' = (y')^2 - y' \quad ; \quad y(0) = 1 \, , \, y'(0) = 2$

$y' = u$ (+)

$y'' = u' = \dfrac{du}{dx} = \dfrac{du}{dy}\underbrace{\dfrac{dy}{dx}}_{y'} = u\dfrac{du}{dy}$ (+)

$y u \dfrac{du}{dy} = u^2 - u \quad ; \quad y\dfrac{du}{dy} = u - 1$

$\dfrac{du}{u-1} = \dfrac{dy}{y} \Rightarrow \int \dfrac{d(u-1)}{(u-1)} = \int \dfrac{dy}{y}$

$\ln|u-1| = \ln y + \ln c_1 = \ln c_1 y$

$\therefore \; u - 1 = c_1 y$ (+)

$\dfrac{dy}{dx} - 1 = c_1 y \Rightarrow \dfrac{dy}{dx} = c_1 y + 1$ (+)

$\int \dfrac{dy}{c_1 y + 1} = \int dx \Rightarrow \dfrac{1}{c_1}\int \dfrac{dy}{y + \frac{1}{c_1}} = \int dx \Rightarrow \int \dfrac{d(y + \frac{1}{c_1})}{(y + \frac{1}{c_1})} = c_1 \int dx$

$\ln\left|y + \frac{1}{c_1}\right| = c_1 x + c_2$

$\therefore \; y + \frac{1}{c_1} = e^{c_1 x + c_2} = c_2 e^{c_1 x}$

$y = c_2 e^{c_1 x} - \frac{1}{c_1}$ (1) General solution

$y(0) = c_2 - \frac{1}{c_1} = 1 \Rightarrow \dfrac{c_2 c_1 - 1}{c_1} = 1$

$y' = c_2 c_1 e^{c_1 x}$

$y'(0) = c_2 c_1 = 2$

$\therefore \; \dfrac{2-1}{c_1} = 1 \Rightarrow c_1 = 1 \Rightarrow c_2 = 2$

$\to$ IVP: $y = 2e^{x} - 1$ (1)