Prof. N. Haidar — FACULTY OF ARTS & SCIENCES , A. U. B.
Sec. 1 & 2 — Math 202 — 23 Nov., 1998
(1-st Semester , 1998-1999)
Time : 1 hour — QUIZ I — (closed book)

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6 I. Find the integrating factor for
$$(2y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy = 0 .$$
Use this integrating factor to solve the DE.

4 II. Given that $y_1 = 1$ is a solution to
$$y' = -x^{-1}(x+1) + x^{-1}(2x+1)\,y - y^2 .$$
Find the general solution of this DE.

4 III. Find the second-order linear DE which is equivalent to the Riccati DE of question II .
Given that $y_1 = e^x$ is a solution to this second-order DE , find its general solution.

5 IV. Solve the IVP
$$(y^2 + 1)\,dx - y\sec^2 x\,dy = 0 \;;\quad y(0) = 0 .$$

6 V. Solve the BVP
$$y''' - 6y'' + 12y' - 8y = 0 \quad ; \quad y(0) = y(1) = 0 ,\ y(-1) = -2 .$$

I ⑥ $(2y^2-6xy)\,dx + (3xy-4x^2)\,dy = 0$

$\frac{\partial M}{\partial y} = 4y-6x$; $\frac{\partial N}{\partial x} = 3y-8x$: $D = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y-6x-3y+8x$

$D = y+2x$ (+) $\therefore \mu \neq 1$

$\frac{D}{-M}$ & $\frac{D}{N}$ will not produce function of x alone or y alone. (+)

Let $\mu(x,y) = x^m y^n$ (1)

$$(\underbrace{2x^m y^{n+2} - 6x^{m+1}y^{n+1}}_{M^+})\,dx + (\underbrace{3x^{m+1}y^{n+1} - 4x^{m+2}y^n}_{N^+})\,dy = 0$$

$$\frac{\partial M^+}{\partial y} = 2x^m(n+2)y^{n+1} - 6x^{m+1}(n+1)y^n$$

$$\frac{\partial N^+}{\partial x} = 3y^{n+1}(m+1)x^m - 4y^n(m+2)x^{m+1}$$

$$D = x^m y^{n+1}[2(n+2)-3(m+1)] - x^{m+1}y^n[6(n+1)-4(m+2)] = 0$$ (+)

$$\Rightarrow \left.\begin{array}{l} 2n+4-3m-3=0 \\ 6n+6-4m-8=0 \end{array}\right\} \begin{array}{l} 2n-3m+1=0 \\ 6n-4m-2=0 \end{array} \times(-3) \Rightarrow \left.\begin{array}{l} -6n+9m-3=0 \\ 6n-4m-2=0 \end{array}\right\}$$

$5m-5=0 \Rightarrow m=1$ (+) $\quad 2n-3+1=0$, $2n=2 \Rightarrow n=1$ (+)

$$(\underbrace{2xy^3 - 6x^2y^2}_{M^+})\,dx + (\underbrace{3x^2y^2 - 4x^3y}_{N^+})\,dy = 0$$ (+)

$$\int M^+\partial x = \int (2xy^3 - 6x^2y^2)\,\partial x = 2y^3\int x\,dx - 6y^2\int x^2\,dx$$

$$= 2y^3\frac{x^2}{2} - 6y^2\frac{x^3}{3} = x^2y^3 - 2x^3y^2$$ (+)

$$I = \frac{\partial}{\partial y}\int M^+\partial x = 3x^2y^2 - 4x^3y$$

$N^+ - I = 0$ (+)

$\therefore g(x,y) = \int M^+\partial x = c \Rightarrow$

$$x^2y^3 - 2x^3y^2 = C$$ (1)

II

$$y' = -x^{-1}(x+1) + x^{-1}(2x+1)\,y - y^2 \quad ; \quad y_1 = 1$$

Riccati D.E. : $P(x) = -x^{-1}(x+1)$, $Q(x) = x^{-1}(2x+1)$, $R(x) = -1$

$n = 2$

Let $y = y_1 + u$ (+)

$$\omega = u^{1-n} = u^{-1}$$

$$\frac{d\omega}{dx} + [Q + 2y_1 R]\,\omega = -R \quad (1)$$

$$\omega' + \left[\frac{2x+1}{x} + 2(1)(-1)\right]\omega = 1$$

$$\omega' + \frac{1}{x}\omega = 1 \quad (1)$$

$$\omega = \mu^{-1}\left[\int \mu\cdot(1)\,dx + c\right]$$

$$\mu = e^{\int \hat{P}(x)\,dx} = e^{\int \frac{dx}{x}} = x$$

$$\omega = x^{-1}\left[\int x\,dx + c\right] = x^{-1}\left[\tfrac{1}{2}x^2 + c\right]$$

$$\omega = \frac{1}{2}x + \frac{c}{x} = \frac{x^2+2c}{2x} = \frac{x^2+c}{2x}$$

$$u = \omega^{-1} = \frac{2x}{c+x^2} \quad (+)$$

$$y = 1 + u = 1 + \frac{2x}{c+x^2} \quad (1)$$

(4)

III. $y' = \underbrace{-x^{-1}(x+1)}_{\tilde{P}(x)} + \underbrace{x^{-1}(2x+1)}_{\tilde{Q}(x)}y \underbrace{-}_{\hat{R}} y^2$; $\omega_1 =$

since $\hat{R}(x) = -1$ let $y = \frac{\omega'}{\omega}$ (1)

$\Downarrow$

$$\omega'' - \hat{Q}(x)\,\omega' - \hat{P}(x)\,\omega = 0$$

$$\omega'' - \left(\frac{2x+1}{x}\right)\omega' + \frac{(x+1)}{x}\,\omega = 0$$

identify this with:

$y'' + P(x)\,y' + Q(x)\,y = 0$; $y_1 = e^x$ (1)

$y'' - \frac{(2x+1)}{x}y' + \frac{(x+1)}{x}y = 0$.

$$y_2 = y_1 \int \frac{e^{-\int P(x)}}{y_1^2(x)}\,dx \quad (+)$$

$$e^{-\int P(x)\,dx} = e^{\int \frac{2x+1}{x}\,dx} = e^{2x+\ln x} = x\,e^{2x}$$

$$y_2 = e^x \int \frac{x e^{2x}}{e^{2x}}\,dx = e^x \int x\,dx = \tfrac{1}{2}x^2 e^x. \quad (+)$$

$\therefore \{y_1, y_2\} = \{e^x, x^2 e^x\}$: F.S.

$$y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 x^2 e^x \quad (1).$$

5

IV
$$(y^2+1)\,dx - y\sec^2 x\,dy = 0 \quad ; \quad y(0)=0$$

$$\frac{dx}{\sec^2 x} = \frac{y}{y^2+1}\,dy$$

$$\int \frac{dx}{\sec^2 x} = \int \frac{y}{(y^2+1)}\,dy \qquad 1$$

$$\int \frac{dx}{\sec^2 x} = \int \cos^2 x\,dx = \frac{1}{2}\int [1+\cos 2x]\,dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \frac{y\,dy}{(y^2+1)} = \frac{1}{2}\int \frac{d(y^2+1)}{(y^2+1)} = \frac{1}{2}\ln|y^2+1|$$

$$\frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\ln|y^2+1| + C$$

$$x + \frac{1}{2}\sin 2x = \ln|y^2+1| + C = \ln(y^2+1) + C$$

$$\ln(y^2+1) = x + \frac{1}{2}\sin 2x - C \qquad 2$$

$$y^2+1 = e^{x+\frac{1}{2}\sin 2x - C}$$

$$y^2 = e^{x+\frac{1}{2}\sin 2x - C} - 1 \qquad 1$$

$$y = \pm\sqrt{e^{x+\frac{1}{2}\sin 2x - C} - 1}$$

$$y(0) = 0$$

$$0 + \frac{1}{2}0 = \frac{1}{2}\ln|1| + C \implies C = 0.$$

$$\longrightarrow \qquad 1$$

$$y = \pm\sqrt{e^{x+\frac{1}{2}\sin 2x} - 1}$$

(6) BVP:

V. $y''' - 6y'' + 12y' - 8y = 0$; $y(0) = y(1) = 0$, $y(-1) = -2$

C.E.: $m^3 - 6m^2 + 12m - 8 = 0$ (1)

$$\begin{array}{r|l} & m^2 - 4m + 4 \\ \hline m-2 & m^3 - 6m^2 + 12m - 8 \\ & m^3 \pm 2m^2 \\ \hline & -4m^2 + 12m - 8 \\ & -4m^2 \mp 8m \\ \hline & 4m - 8 \\ & 4m - 8 \\ \hline \end{array}$$ (+)

$(m-2)(m^2 - 4m + 4) = 0$

$(m-2)(m-2)^2 = 0$

$(m-2)^3 = 0$

$\therefore m_1 = m_2 = m_3 = 2$ (1)

$\{ y_1 = e^{2x}, y_2 = xe^{2x}, y_3 = x^2 e^{2x} \}$: F.S.

$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$ = $P_2(x)\, e^{2x}$ (1)

$y(0) = 0 \Rightarrow c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$ (+)

$\longrightarrow$

$y = c_2 x e^{2x} + c_3 x^2 e^{2x}$

$y(1) = c_2 e^2 + c_3 e^2 = 0 \Rightarrow c_2 + c_3 = 0 \Rightarrow c_3 = -c_2$

(+)

$y(-1) = c_2(-1)e^{-2} + c_3(-1)^2 e^{-2} = -2$

$-c_2 + c_3 = -2e^2 \Rightarrow$

$\cancel{c_2} + c_3 = 0$

$-\cancel{c_2} + c_3 = -2e^2$

$2c_3 = -2e^2$

$c_3 = -e^2$

$c_2 = e^2$ (+)

$\therefore y = e^2 x e^{2x} - e^2 x^2 e^{2x}$

IVP

$y = x e^{2x+2} - x^2 e^{2x+2} = x(1-x)\, e^{2x+2}$ (1)