American University of Beirut

MATH 202

Differential Equations

Spring 2009

quiz # 2 - solution

Exercise 1 Find the general solution of the given differential equation (*do not find the constants)*

a) $y''' - y'' = 6$

$y = \underbrace{c_0 + c_1 x + c_2 e^x}_{y_c} + \underbrace{c_3 x^2}_{y_p}$

b) $y'' - 4y' + 5y = e^{-x} + 2\cos(2x)$

$y = \underbrace{e^{2x}(c_0 \cos x + c_1 \sin x)}_{y_c} + \underbrace{c_2 e^{-x}}_{y_{p1}} + \underbrace{c_3 \cos(2x) + c_4 \sin(2x)}_{y_{p2}}$

c) $y^{(4)} - 2y'' + y = 1 + x - xe^x + \sin x$

$y = \underbrace{c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}}_{y_c} + \underbrace{c_5 + c_6 x}_{y_{p1}} + \underbrace{(c_7 x^2 + c_8 x^3) e^x}_{y_{p2}} + \underbrace{c_9 \sin x + c_{10} \cos x}_{y_{p3}}$

Exercise 2 Find the general solution of $y'' - y = \dfrac{2e^x}{e^x + e^{-x}}$

$y_c = c_1 e^x + c_2 e^{-x}$; $W = -2$,

$$c_1' = \frac{1}{e^x + e^{-x}} = \frac{e^x}{1 + e^{2x}}, \text{ and } c_1 = \int \frac{e^x}{1 + e^{2x}}\, dx = \tan^{-1}(e^x)$$

$$c_2' = -\frac{e^{2x}}{e^x + e^{-x}}, \text{ and } c_2 = -\int \frac{e^{2x}}{e^x + e^{-x}}\, dx = -\int \frac{u^2}{u + \frac{1}{u}} \frac{du}{u} = -\int \frac{u^2}{1 + u^2}\, du$$

$$= -\int \left(1 - \frac{1}{1 + u^2}\right) du = \tan^{-1}(u) - u = \tan^{-1}(e^x) - e^x$$

hence $y_p = (e^x + e^{-x}) \tan^{-1}(e^x) - 1$

and the general solution is: $y = y_c + y_p$

Exercise 3 Consider the differential equation

$$(E): \ x^2 y'' - (x^2 + 2x) y' + (x + 2) y = x^3$$

a) check that $y_1 = x$ is solution of (E_0): **obvious by direct substitution**.

b) let $y_2 = xu(x)$. Show that $u(x)$ satisfies a first order linear differential equation; find $u(x)$ then find the general solution of (E) on $(0, \infty)$.

$y_2' = u(x) + xu'(x)$, $y_2'' = 2u'(x) + xu''(x)$, and substituting in (E) yields: $u''(x) - u'(x) = 1$. Taking $v = u'$, then $v' - v = 1$, and then $v(x) = e^x - 1$, and $u(x) = e^x - x$.

The general solution of (E) is then: $y = \underbrace{c_1 x + c_2 x e^x}_{y_c} + \underbrace{-x^2}_{y_p}$

Exercise 4 Use the substitution $x = e^t$ to solve the Cauchy-Euler differential equation

$$x^2y'' - xy' + y = x(\ln x)^2$$

on $(0, \infty)$.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Substituting in the equation yields: $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = t^2e^t$, and the solution is

$$y = \underbrace{c_1e^t + c_2te^t}_{y_c} + \underbrace{(c_3t^2 + c_4t^3 + c_5t^4)e^t}_{y_p}$$

hence $y = c_1x + c_2x\ln x + c_3x(\ln x)^2 + c_4x(\ln x)^3 + c_5x(\ln x)^4$

Exercise 5 Find two power series solutions of the differential equation $y'' - xy' + y = 0$ about the ordinary point $x = 0$. Give the radius of convergence.

Let $y = \sum_{n=0}^{\infty} c_nx^n$ be a series solution of the differential equation. Deriving and substituting in the equation yields the following:

$$\begin{cases} c_0 + 2c_2 = 0 & (1) \\ (n+2)(n+1)c_{n+2} - (n-1)c_n = 0 \quad n \geq 1 & (2) \end{cases}$$

equation (1) implies $c_2 = -\frac{1}{2}\,c_0$

$c_1 \in \mathbb{R}$, and from equation (2), we find that $c_3 = c_5 = c_7 = \ldots = c_{2n+1} = \ldots = 0$, then $y_1 = x$ is a solution.

form equation (2), $c_{2n} = \dfrac{2n-3}{2n(2n-1)}\,c_{2n-2}$; solving the recurrence yields $c_{2n} = -\dfrac{2n-3}{2^n\,n!}\,c_0$, and $y_2 = \sum_{n=0}^{\infty} c_{2n}x^{2n}$ is the other solution of the differential equation. The radius of convergence is $R = \infty$ (by ratio test).

The general solution is then: $y = c_0\sum_{n=0}^{\infty}\dfrac{2n-3}{2^n\,n!}x^{2n} + c_1x$