MATH 202 Spring 2005

Final exam solutions, Sections 1-4 (Professor Kamal Khuri-Makdisi)

① $y^2 \ln y \, dy = \dfrac{(3x^2+1)\,dx}{\sqrt{x^3+x+1}}$

$$\int y^2 \ln y \, dy = \int \frac{3x^2+1}{\sqrt{x^3+x+1}}\,dx + C$$

by Parts

$$\int y^2 \ln y \, dy = y^3 \ln y - \int y \, d(y^2 \ln y) \cancel{- y^3 \ln y c}$$

$$d(y^2 \ln y) = \left(2y \ln y + \frac{y^2}{y}\right)dy = (2y \ln y + y)\,dy$$

$$\int y^2 \ln y \, dy = y^3 \ln y - \int (2y^2 \ln y + y^2)\,dy$$

$$= y^3 \ln y - 2\int y^2 \ln y \, dy - \frac{y^3}{3}$$

$$3\int y^2 \ln y \, dy = y^3 \ln y - \frac{y^3}{3}$$

$$\int y^2 \ln y \, dy = \frac{y^3 \ln y}{3} - \frac{y^3}{9}$$

$$\int \frac{3x^2+1}{\sqrt{x^3+x+1}}\,dx = \int \frac{d(x^3+x+1)}{\sqrt{x^3+x+1}} = 2\sqrt{x^3+x+1}$$

General Solution

$$\frac{y^3 \ln y}{3} - \frac{y^3}{9} = 2\sqrt{x^3+x+1} + C$$

b) let $u = y^{1-\frac{3}{4}} = y^{+\frac{1}{4}}$ $\quad u' = +\frac{1}{4} y^{-\frac{3}{4}} y'$

$$\frac{1}{4} y^{-\frac{3}{4}} y' + \frac{8}{x} \times \frac{1}{4} y^{-\frac{3}{4}} y = \frac{\cancel{4}\tan x}{x^2} \times \frac{1}{\cancel{4}}$$

$$u' + \frac{2}{x} u = \frac{\tan x}{x^2}$$

; let $g = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

$$x^2 u' + 2xu = \tan x$$

$(x^2 u)'$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x}\,dx = -\ln(\cos x)$$

$$(x^2 u) = -\ln(\cos x) + C$$

$$\boxed{x^2 y^{\frac{1}{4}} = -\ln|\cos x| + C}$$

$$\boxed{y = \left(\cancel{\frac{-\ln|\cos x|}{x^2}} + \frac{C}{x^2}\right)^4}$$

② Let $\tilde{M} = 4\mu y^3 + 3x^2y^2\mu$ and $\tilde{N} = 3xy^2\mu + x^3y\mu$

$\tilde{M}_y = 12\mu y^2 + 4\mu_y y^3 + 6x^2y\mu + 3x^2y^2\mu_y$

$\tilde{N}_x = 3y^2\mu + 3xy^2\mu_x + 3x^2y\mu + x^3y\mu_x$

Need $\tilde{M}_y = \tilde{N}_x$ Let $\mu = \mu(x)$ $\mu_y = 0$

$12\mu y^2 + 6x^2y\mu = 3y^2\mu + 3xy^2\mu_x + 3x^2y\mu + x^3y\mu_x$

$9\mu y^2 + 3x^2y\mu = \mu_x(3xy^2 + x^3y)$

$\mu\cancel{y}(9y + 3x^2) = \mu_x x\cancel{y}(3y + x^2)$

$\mu(9y + 3x^2) = \frac{x}{3}\mu_x(9y + 3x^2)$

$x\frac{d\mu}{dx} = 3\mu$ $\cancel{\frac{d\mu}{\mu} = 3dx \quad \ln\mu = 3x \quad \mu = e^{3x}}$

$\cancel{\tilde{M} = 4e^{3x}y^3 + 3x^2y^2e^{3x} \quad \tilde{N} = 3xy^2e^{3x} + x^3ye^{3x}}$

$\cancel{f(x,y) = \frac{4}{3}e^{3x} \quad f(x,y) = e^{3x}y^4 + x^2y^3e^{3x} + h(x)}$

$\cancel{\underbrace{\frac{\partial f}{\partial x}}_{\tilde{N}} = 3e^{3x}y^4 + 2xy^3e^{3x} + 3x^2y^3e^{3x} + h'(x)}$

$\cancel{3xy^2e^{3x} + x^3ye^{3x} = 3e^{3x}y^4 +}$

$\frac{d\mu}{\mu} = \frac{3dx}{x}$ $\ln\mu = 3\ln x$ $\boxed{\mu = x^3}$ 12

$\boxed{\tilde{M} = 4x^3y^3 + 3x^5y^2 \quad \tilde{N} = 3x^4y^2 + x^6y}$ for exact eqn. $\tilde{M}dx + \tilde{N}dy = 0$ gotten after mult. by μ

$f(x,y) = x^4y^3 + \frac{1}{2}x^6y^2 + h(y)$

$\underbrace{\frac{\partial f}{\partial y}}_{\tilde{N}} = 3x^4y^2 + x^6y + h'(y)$ thus $h'(y) = 0$

take $h(y) = 0$

Equation becomes $df = 0$ ($\tilde{M}dx + \tilde{N}dy = 0$)

General Solution $f = c$

$\boxed{x^4y^3 + \frac{1}{2}x^6y^2 = C}$ $\cancel{3x^4y^2 + x^6y = C}$

(3) $y'' - 2y' + y = 0$ Charact. Eqn: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_1 = m_2 = 1$ $\boxed{y_c = c_1 e^x + c_2 x e^x}$

$L[y_{p_1}] = x^2$; $L[x^2] = 2 - 4x + x^2$; $L[x] = -2 + x$; $L[1] = 1$

$A = 1$; $-4A + B = 0$ $B = 4$; $2A - 2B + C = 0$ $C = 8 - 2 = 6$

$\boxed{y_{p_1} = x^2 + 4x + 6}$

Variation of parameters $y_1 = e^x$ $y_2 = xe^x$ $f = \frac{e^x}{x^2+1}$

$$\begin{cases} u_1' e^x + u_2' x e^x = 0 \\ u_1' e^x + u_2' e^x + u_2' x e^x = \frac{e^x}{x^2+1} \end{cases}$$

$u_2' e^x = \frac{e^x}{x^2+1}$ $u_2' = \frac{1}{x^2+1}$ $u_2 = \text{Arctan}\, x$

$u_1' = -u_2' x$ $u_1' = \frac{-x}{x^2+1}$ $u_1 = \frac{1}{2}\int \frac{-d(x^2+1)}{x^2+1}$

$u_1 = -\frac{1}{2} \ln(x^2+1)$

General Solution:

$$y = \left(c_1 - \frac{1}{2}\ln(x^2+1)\right)e^x + (c_2 + \text{Arctan}\, x)\, x e^x + x^2 + 4x + 6$$

12

(4) Characteristic Equ: $r(r-1)(r-2)-8r+8=0$

$r(r-1)(r-2)-8(r-1)=0$

$(r-1)(r^2-2r-8)=0$ $\quad \Delta' = 1+8=9$ $\quad r_2 = 1-3 = -2$

$r_3 = 1+3 = 4$

$r_1 = 1 \quad r_2 = -2 \quad r_3 = 4$

$y = c_1 x + c_2 x^{-2} + c_3 x^4$ $\quad y' = c_1 - 2c_2 x^{-3} + 4c_3 x^3$; $y'' = 6c_2 x^{-4} + 12 c_3 x^2$

$y(1)=1$; $\quad 1 = c_1 + c_2 + c_3$

$y'(1)=7$; $\quad 7 = c_1 - 2c_2 + 4c_3$

$y''(1)=0$; $\quad 0 = 6c_2 + 12c_3$ $\quad \boxed{c_2 = -2c_3}$ $\quad \boxed{-2c_2 = +4c_3}$

$1 = c_1 - c_3$ $\quad 9c_3 = 6$ $\quad \boxed{c_3 = \frac{2}{3}}$

$7 = c_1 + 8c_3$

$c_1 = 1 + c_3 = 1 + \frac{2}{3} = \frac{5}{3}$

$\boxed{c_1 = \frac{5}{3}}$

$\boxed{c_2 = -\frac{4}{3}}$

$$\boxed{y = \frac{5}{3}x - \frac{4}{3}x^{-2} + \frac{2}{3}x^4}$$

12

(5) $y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \quad y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

$\downarrow n = K$ $\qquad$ $\downarrow n+2 = K-2$

$K = n+4 \qquad K: 4 \to \infty$

$n = K-4$

$$\sum_{K=0}^{\infty} K(K-1) c_K x^{K-2} + \sum_{K=4}^{\infty} c_{K-4} x^{K-2} = 0$$

Initial Identities; $K=0$; $\quad 0 c_0 = 0$] arbitrary

$K=1$; $\quad 0 c_1 = 0$]

$K=2$; $\quad 2c_2 = 0 \quad \boxed{c_2 = 0}$

$K=3$; $\quad 6c_3 = 0 \quad \boxed{c_3 = 0}$

Recurrence $K \geqslant 4$.

$$K(K-1) c_K + c_{K-4} = 0 \qquad c_K = \frac{-1}{K(K-1)} c_{K-4}$$

$$c_0 \xrightarrow{\frac{-1}{4(3)}} c_4 \xrightarrow{\frac{-1}{(8)(7)}} c_8 \xrightarrow{\frac{-1}{(12)(11)}} c_{12} \; --- \xrightarrow{\frac{-1}{(4\ell)(4\ell-1)}} c_{4\ell} \to --$$

$$\boxed{c_{4\ell} = \frac{(-1)^\ell}{\prod_{j=1}^{\ell} (4j)(4j-1)} c_0}$$

$$c_1 \xrightarrow{\frac{-1}{5(4)}} c_5 \xrightarrow{\frac{-1}{(9)(8)}} c_9 \xrightarrow{\frac{-1}{(13)(12)}} c_{13} \; -- \xrightarrow{\frac{-1}{(4\ell+1)(4\ell)}} c_{4\ell+1} \to --$$

$$\boxed{c_{4\ell+1} = \frac{(-1)^\ell}{\prod_{j=1}^{\ell} (4j)(4j+1)} c_1}$$

$$\boxed{c_{4\ell+2} = 0 \quad c_{4\ell+3} = 0 \quad \text{for all } \ell}$$

General Solution:

$$y = c_0 \left[1 + \frac{-1}{(4)(3)} x^4 + \frac{(-1)(-1)}{(4)(3)(8)(7)} x^8 + \frac{(-1)(-1)(-1)}{(4)(3)(8)(7)(12)(11)} x^{12} + \dots + \frac{(-1)^\ell}{\prod_{j=1}^{\ell} (4j)(4j-1)} x^{4\ell} + \dots \right]$$

$$+ c_1 \left[x + \frac{-1}{(5)(4)} x^5 + \frac{(-1)(-1)}{(5)(4)(9)(8)} x^9 + \frac{(-1)(-1)(-1)}{(5)(4)(9)(8)(12)(13)} x^{13} + \dots + \frac{(-1)^\ell}{\prod_{j=1}^{\ell} (4j)(4j+1)} x^{4\ell+1} + \dots \right]$$

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(6) $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

$$3x^2 y'' = \sum_{n=0}^{\infty} 3c_n (n+r)(n+r-1) x^{n+r}$$

$$2xy' = \sum_{n=0}^{\infty} 2c_n (n+r) x^{n+r}$$

$$-xy = -\sum_{n=0}^{\infty} c_n x^{n+r+1}$$

$$\sum_{n=0}^{\infty} 3c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 2c_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\cancel{\sum_{n=0}^{\infty} c_n \left[3(n+r)^2 - 3(n+r)\right.}$$

$$\sum_{n=0}^{\infty} c_n (n+r)\left(3(n+r) - 3 + 2\right) x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} 3c_n (n+r)\left(n+r - \tfrac{1}{3}\right) x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$\downarrow n = k$ $\quad\quad \downarrow$ $n+1 = k$, $k: 1 \to \infty$, $n = k-1$

$$\sum_{k=0}^{\infty} 3c_k (k+r)\left(k+r - \tfrac{1}{3}\right) x^{k+r} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r} = 0$$

<u>Initial Id</u> : $c_0 \neq 0$; $\quad k = 0$, $3r\left(r - \frac{1}{3}\right)c_0 = 0$

$r_2 = 0 \quad \boxed{r_1 = \frac{1}{3}}$

<u>Recurrence</u>: $r_1 = \frac{1}{3}$; $k \geq 1$

$$c_k (3k+1)k - c_{k-1} = 0$$

$$c_k = \frac{1}{k(3k+1)} c_{k-1}$$

USE FOR ANSWER

$$c_0 \xrightarrow{\frac{1}{(1)(4)}} c_1 \xrightarrow{\frac{1}{(2)(7)}} c_2 \xrightarrow{\frac{1}{(3)(10)}} c_3 \; --- \; \xrightarrow{\frac{1}{(\ell)(3\ell+1)}} c_\ell \rightarrow -$$

$$c_\ell = \frac{1}{(\ell)! \prod_{j=1}^{\ell}(3j+1)} c_0$$

~~Gen~~ Solution: $r_1 = \frac{1}{3}$

$$y = c_0 x^{\frac{1}{3}} \Big[1 + \frac{1}{(1)(4)} x + \frac{1}{(1)(2)(4)(7)} x^2 + \frac{1}{3!\,(4)(7)(10)} x^3 + + --- + \frac{1}{(\ell)! \prod_{j=1}^{\ell}(3j+1)} x^\ell + --- \Big]$$

$$= c_0 \Big[x^{1/3} + \frac{1}{1\cdot 4} x^{1+1/3} + \frac{1}{2!\,4\cdot 7} x^{2+1/3} + \frac{1}{3!\,4\cdot 7\cdot 10} x^{3+1/3} + --- + \frac{1}{\ell!\, \prod_{j=1}^{\ell}(3j+1)} x^{\ell+1/3} + \dots \Big]$$

12

(7) Let $\mathcal{L}\{y\} = Y$ $\quad \mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0)$

$$= s^2 Y - 2$$

$\mathcal{L}\{e^t\} = \frac{1}{s-1}$ $\quad \mathcal{L}\{\delta(t-3)\} = e^{-3s}$

$$s^2 Y - 2 - 4Y = \frac{1}{s-1} + e^{-3s}$$

$$Y(s-2)(s+2) = \frac{1}{s-1} + e^{-3s} + 2$$

$$Y = \frac{1}{(s-1)(s-2)(s+2)} + \frac{e^{-3s}}{(s-2)(s+2)} + \frac{2}{(s-2)(s+2)}$$

$\frac{1}{(s-1)(s-2)(s+2)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$; Cover-Up Method

$A = \frac{1}{(-1)(3)} = \frac{-1}{3}$; $B = \frac{1}{(1)(4)} = \frac{1}{4}$; $C = \frac{1}{(-3)(-4)} = \frac{1}{12}$

$\frac{1}{(s-2)(s+2)} = \frac{D}{(s-2)} + \frac{E}{(s+2)}$; Cover-Up Method

$D = \frac{1}{4}$; $E = \frac{-1}{4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s+2)}\right\} = \frac{-1}{3} e^t + \frac{1}{4} e^{2t} + \frac{1}{12} e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s+2)}\right\} = \frac{1}{2} e^{2t} - \frac{1}{2} e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-2)(s+2)}\right\} = \frac{1}{4} u(t-3) e^{2(t-3)} - \frac{1}{4} u(t-3) e^{-2(t-3)}$$

$$\boxed{y(t) = -\frac{1}{3} e^t + \frac{3}{4} e^{2t} - \frac{5}{12} e^{-2t} + \frac{1}{4} u(t-3) e^{2(t-3)} - \frac{1}{4} u(t-3) e^{-2(t-3)}}$$

$\frac{10}{16}$

8 a) $\frac{s+3}{s^2+4s+13} = \frac{s+3}{s^2+4s+4+9} = \frac{s+3}{(s+2)^2+9}$

$= \frac{s+2}{(s+2)^2+9} + \frac{1}{3}\,\frac{3}{(s+2)^2+9}$

$\cancel{\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} = e^{-2t}}$

$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4s+13}\right\} = e^{-2t}\cos(3t) + \frac{1}{3}e^{-2t}\sin(3t)$ ✓

b) $f(t) = 1 - u(t-2) + t\,(u(t-2))$

$= 1 + (t-1)u(t-2) = 1 + (t-2)u(t-2) + u(t-2)$

$\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$ ✓

c) $h(t) = \int_{\tau=0}^{t} \tau^{202}(t-\tau)^{2005}\,d\tau = f(t) * g(t)$

where $f(t) = t^{202}$ and $g(t) = t^{2005}$

$\mathcal{L}\{h(t)\} = \mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}$

$\mathcal{L}\{f(t)\} = \frac{(202)!}{s^{203}} \qquad \mathcal{L}\{g(t)\} = \frac{(2005)!}{s^{2006}}$

$H(s) = \mathcal{L}\{h(t)\} = \frac{(202)!}{s^{203}} * \frac{(2005)!}{s^{2006}}$

$= \frac{(202)!\,(2005)!}{s^{2209}}$

ANSWER

$\boxed{\frac{(202)!\,(2005)!}{(2208)!}\,t^{2008}}$

$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{(202)!\,(2005)!}{(2208)!}\,\mathcal{L}^{-1}\left\{\frac{(2208)!}{s^{2209}}\right\} =$

12

(9) $\vec{x}\,' = \begin{pmatrix} 0 & 0 & 2 \\ 2 & -1 & 4 \\ 1 & 0 & 1 \end{pmatrix}$ First $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 2 & -1-\lambda & 4 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$+\lambda(+1+\lambda)(1-\lambda) + 2(1+\lambda) = 0$$

$$(1+\lambda)(-\lambda^2+\lambda+2) = 0 \qquad (1+\lambda)\underbrace{(\lambda^2-\lambda-2)}_{(\lambda+1)(\lambda-2)} = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-2) = 0$$

$$\lambda_1 = \lambda_2 = -1; \quad \lambda_3 = 2$$

$$\boxed{\vec{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}}$$

$(A - \lambda I)\vec{K} = \vec{0} \qquad \lambda_1 = -1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$K_1 = -2K_3$$

$$\vec{X}_1 = e^{-t}\begin{pmatrix} -2K_3 \\ 0 \\ K_3 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ K_2 \\ 0 \end{pmatrix}$$

let $K_3 = 1$, $K_2 = 0$ $\quad K_2 = 1$, $K_3 = 0$ will appear in la... cons...

$$\boxed{\vec{X}_1 = e^{-t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \qquad \boxed{\vec{X}_2 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda_3 = 2$

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 2 & -3 & 4 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} K_1 = K_3 \\ K_2 = 2K_3 \end{array}$$

Let K_3 here = free

$$\vec{x_3} = e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

General Solution

$$\cancel{\vec{x} = c_1 e^{t}} \quad \vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$$

$$\vec{x} = c_1 e^{-t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

12

(10) $\vec{x}' = \begin{pmatrix} -2 & 6 \\ -3 & 4 \end{pmatrix} \vec{x}$

Eigenvalues $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 6 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(4-\lambda) + 18 = 0$$

$$-8 + 2\lambda - 4\lambda + \lambda^2 + 18 = 0$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\Delta' = 1 - 10 = -9 = 9i^2$$

$$\lambda_1 = \frac{1-3i}{1} = 1-3i \quad \lambda_2 = 1+3i$$

$(A - \lambda I)\vec{K} = 0$ Eigenvectors $\lambda = 1+3i$ $\boxed{\vec{K} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}$

$$\left(\begin{array}{cc|c} -3-3i & 6 & 0 \\ -3 & 3-3i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 3+3i & -6 & 0 \\ -3 & 3-3i & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1+i & 0 \\ -1 & 1-i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

- $k_1 = (1-i)k_2$ $\boxed{\text{take } k_2 = 1}$ $\boxed{e^{(1+3i)t} = e^t(\cos 3t + i \sin 3t)}$

$\cancel{\vec{x}_1 = }$ $\vec{x}_1 = e^t(\cos 3t + i \sin 3t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\boxed{\vec{K} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \text{ eigenvector for } 1+3i}$ $+ i e^t(\cos 3t + i \sin 3t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\vec{x}_1 = e^t \cos 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
$$+ i\left[e^t \sin 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^t \cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$$

Let $\vec{x}_1 = \vec{y} + i\vec{z}$

Solution is $\vec{x} = c_1 \vec{y} + c_2 \vec{z}$

$$\boxed{\begin{aligned}\vec{x} &= c_1 \left[e^t \cos 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &+ c_2 \left[e^t \cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + e^t \sin 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]\end{aligned}}$$

mention $\vec{x}_2 = \vec{y} - i\vec{z}$ so $\vec{x} = c_1\vec{y} + c_2\vec{z}$

10

(11) a) $\dfrac{b_1 x^2 + b_2 x + b_3}{(x^2+2x+5)(x+1)} = \dfrac{a_1 x + a_2}{x^2+2x+5} + \dfrac{a_3}{x+1}$

$$\frac{b_1 x^2 + b_2 x + b_3}{(x^2+2x+5)(x+1)} = \frac{(a_1 x + a_2)(x+1) + a_3 x^2 + 2a_3 x + 5a_3}{(x^2+2x+5)(x+1)}$$

$$b_1 x^2 + b_2 x + b_3 = a_1 x^2 + a_1 x + a_2 x + a_2 + a_3 x^2 + 2a_3 x + 5a_3$$

$$b_1 x^2 + b_2 x + b_3 = x^2(a_1 + a_3) + x(a_1 + a_2 + 2a_3) + a_2 + 5a_3$$

$\cancel{a_1 + a_3}$

$$\begin{cases} a_1 + 0a_2 + a_3 = b_1 \\ a_1 + a_2 + 2a_3 = b_2 \\ 0a_1 + a_2 + 5a_3 = b_3 \end{cases}$$

Thus $\cancel{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 5 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

b) $\det(A) = ?$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 5 \end{vmatrix} = 1(5-2) - 0 + 1 = 3 + 1 \cancel{\neq 0} = 4 \neq 0$$

Since $\det A \neq 0$ then A is invertible

c) $$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -4 & -1 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}}$$

d) $A \times A^{-1} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ Multiply both Sides by A^{-1}

$$I \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\boxed{\begin{aligned} a_1 &= \frac{3}{4} b_1 + \frac{1}{4} b_2 - \frac{1}{4} b_3 \\ a_2 &= -\frac{5}{4} b_1 + \frac{5}{4} b_2 - \frac{1}{4} b_3 \\ a_3 &= \frac{1}{4} b_1 - \frac{1}{4} b_2 + \frac{1}{4} b_3 \end{aligned}}$$

12