Not To Be Taken Out
Room

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Reserve Reading Room

Professor Makdisi

Math 202 — Spring 2006
Differential Equations, sections 1–4
Final exam, June 7 — Duration: 2 hours

JUST A MINUTE — PLEASE READ THE INSTRUCTIONS BELOW FIRST

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1. **Write your name, AUB ID number, and section number ON THE FRONT COVER OF YOUR AUB EXAMINATION BOOKLET.**
To remind you, the sections are as follows:

Section 1	Section 2	Section 3	Section 4
Recitation Th 12:30	Recitation Th 2	Recitation F 10	Recitation F 9
Ms. Jaber	Ms. Jaber	Prof. Makdisi	Prof. Makdisi

2. **You may work on the problems in ANY ORDER in your exam booklet, but please make it clear which problem you are solving on any given page. In particular, PLEASE INDICATE IF THE SOLUTION TO A PROBLEM IS CONTINUED ON A LATER PAGE.**

3. **Do as much of the exam as you can, and budget your time carefully. THE PROBLEMS ARE ARRANGED ACCORDING TO THE CHAPTERS COVERED IN THE BOOK (i.e., chapters 2, 4, 6, 7, 8); just because a problem comes later does NOT mean that it is harder than the earlier problems.**

4. **Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points, and there are 10 problems for a TOTAL 120 points.**

5. **If you cannot do a certain integral, just leave it as an integral in your solution for partial credit on the rest of the problem.**

6. **No calculators, books, or notes allowed. Turn off and put away any cell phones or beepers.**

GOOD LUCK!

(Remember, each problem is worth 12 points for a total of 120)

1. Find the general solution of

$$(x^2+4)y' - 2xy = (x^2+4)(x^3+1).$$

2. Using an integrating factor μ, find the general solution of

$$(2x+1)\cos y\,dx + \Big((x^2+x)\sin y + \tan y\Big)dy = 0.$$

[No "canned" formulas allowed for μ!]

3. Find the general solution of

$$y''' - 4y'' - 2y' + 20y = e^{-2x}.$$

* continued on reverse side *

4. Find the general solution of

$$4x^2y'' + y = \frac{x^{1/2}}{1+\ln x}.$$

[Be careful with variation of parameters. You may write down a system of equations for u_1' and u_2' directly, but you may not use a "canned" formula for the solution of these equations.]

5. Using a series centered at $x = 0$, find the general solution of

$$(1+x^2)y'' - 6xy' + 10y = 0.$$

[Hint: the two fundamental solutions y_1 and y_2 are both polynomials.]

6. Find m so that the substitution $y = x^m u$ reduces the following equation to a Bessel DE, and use this to find the general solution of the following equation:

$$x^2y'' - 5xy' + \left(x^2 + \frac{11}{4}\right)y = 0.$$

7. Use Laplace transforms to solve the Volterra integral equation

$$f(t) - \int_{\tau=0}^{t} (t-\tau)e^{-2(t-\tau)} f(\tau)\, d\tau = e^t.$$

8. Use Laplace transforms to solve

$$y'' + 9y = \begin{cases} 1, & \text{if } t < 3 \\ t-2, & \text{if } t \geq 3, \end{cases}; \qquad y(0) = 1, \quad y'(0) = 0.$$

[Suggestion: define $g(t) = \begin{cases} 1, & \text{if } t < 3 \\ t-2, & \text{if } t \geq 3, \end{cases}$; **and first compute the Laplace transform** $G(s) = \mathcal{L}\{g(t)\}$.**]**

9. Find the general solution of

$$\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 4 & 0 \\ 0 & -1 & -5 \\ 0 & 1 & 1 \end{pmatrix} \vec{\mathbf{x}}.$$

[Please leave the answer in COMPLEX form; exceptionally, there are no points for converting the answer to real numbers in this exercise.]

10. Find the general solution of

$$\vec{\mathbf{x}}' = \begin{pmatrix} 0 & -9 \\ 1 & 6 \end{pmatrix} \vec{\mathbf{x}}.$$

[Reminder: $te^{\lambda t}\vec{\mathbf{L}} + e^{\lambda t}\vec{\mathbf{M}}$ **is a particular solution if** $(A - \lambda I)\vec{\mathbf{L}} = \vec{\mathbf{0}}$ **and** $(A-\lambda I)\vec{\mathbf{M}} = \vec{\mathbf{L}}$.**]**

(AW)

Math 202 – Spring 2006 solutions
Professor Khuri-Makdisi

$\boxed{1.}$ Normalize this linear equation:

$$y' - \frac{2x}{x^2+4}\, y = x^3+1 \qquad \text{so } y' + Py = Q, \quad P = \frac{-2x}{x^2+4}$$

then the integrating factor is $\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{-2x\,dx}{x^2+4}} = e^{-\int \frac{d(x^2+4)}{x^2+4}}$

so $\mu = e^{-\ln(x^2+4)}$ & $\boxed{\mu = \frac{1}{x^2+4}}$.

Multiply the normalized equation by μ to get:

$$\frac{y'}{x^2+4} - \frac{2x}{(x^2+4)^2}\, y = \frac{x^3+1}{x^2+4},$$

$$\Leftrightarrow \left(\frac{y}{x^2+4}\right)' = \frac{x^3+1}{x^2+4},$$

scratch:

$$\begin{array}{r|l} & x \\ x^2+4 & x^3 + \quad 1 \\ & \ominus x^3 \ominus 4x \\ & \quad -4x+1 \end{array}$$

$$\Leftrightarrow \frac{y}{x^2+4} = \int\left(\frac{x^3+1}{x^2+4}\right)dx + C$$

$$= \int\left(x + \frac{-4x+1}{x^2+4}\right)dx + C$$

$$= \frac{x^2}{2} - 2\int\frac{2x\,dx}{x^2+4} + \int\frac{dx}{x^2+4} + C$$

$$\therefore \boxed{\frac{y}{x^2+4} = \frac{x^2}{2} - 2\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

[here $\int\frac{dx}{x^2+4} = \int\frac{2\,du}{4u^2+4} = \frac{1}{2}\int\frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}u$
put $x = 2u$
$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$]

$\boxed{2.}$ multiply by μ:

$$\underbrace{(2x+1)\cos y \cdot \mu}_{\text{this is } M} \cdot dx + \underbrace{\left((x^2+x)\sin y + \tan y\right)\cdot \mu}_{\text{this is } N} \cdot dy = 0$$

we want $M_y = N_x$, i.e.

$$-(2x+1)\sin y \; \mu + (2x+1)\cos y \cdot \frac{\partial \mu}{\partial y} = (2x+1)\sin y \cdot \mu + \begin{pmatrix}(x^2+x)\sin y \\ +\tan y\end{pmatrix} \frac{\partial \mu}{\partial x}$$

Take $\mu = \mu(y)$ so $\frac{\partial \mu}{\partial x} = 0$; also cancel $(2x+1)$

so $$-\sin y \cdot \mu + \cos y \cdot \frac{d\mu}{dy} = \sin y \cdot \mu \quad (+0)$$

rearrange: $$\frac{d\mu}{\mu} = \frac{2\sin y \, dy}{\cos y} = -\frac{2\, d(\cos y)}{\cos y}$$

integrate (we only need one μ): $$\ln \mu = -2\ln(\cos y) \implies \boxed{\mu = \frac{1}{\cos^2 y}}$$

the DE becomes

$$\underbrace{\frac{(2x+1)}{\cos y}}_{\text{this is } f_x} dx + \underbrace{\left(\frac{(x^2+x)\sin y + \tan y}{\cos^2 y}\right)}_{\text{this is } f_y} dy = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f = \frac{x^2+x}{\cos y} + g(y) \implies \frac{\cancel{(x^2+x)\sin y}}{\cancel{\cos^2 y}} + \frac{dg}{dy} = \frac{\cancel{(x^2+x)\sin y} + \tan y}{\cos^2 y}$$

$$\implies \frac{dg}{dy} = \frac{\tan y}{\cos^2 y}, \quad g = \int \frac{\tan y \, dy}{\cos^2 y} = \int \tan y \; d(\tan y) = \frac{1}{2}\tan^2 y$$

So the DE is $df = 0$ where

$$f(x,y) = \frac{x^2+x}{\cos y} + \frac{1}{2}\tan^2 y$$

& the soln. is $f(x,y) = C$, i.e. $\boxed{\frac{x^2+x}{\cos y} + \frac{1}{2}\tan^2 y = C}$

$\boxed{3}$ the aux. eqn is $m^3 - 4m^2 - 2m + 20 = 0$

try factors of 20 as roots; we find $(-2)^3 - 4(-2)^2 - 2(-2) + 20$
i.e. $\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$
but stop when you find a root

$= -8 - 16 + 4 + 20 = 0$

$\therefore m = -2$ is a root & $(m+2)$ is a factor

$$\begin{array}{r|l} & m^2 - 6m + 10 \\ \hline m+2 & m^3 - 4m^2 - 2m + 20 \\ & \overset{\ominus}{m^3} + 2m^2 \\ \hline & -6m^2 - 2m + 20 \\ & \overset{\oplus}{-}6m^2 \overset{\oplus}{-} 12m \\ \hline & 10m + 20 \\ & \ominus\ 10m \overset{\ominus}{+} 20 \\ \hline & 0 \end{array}$$

$\Rightarrow m^3 - 4m^2 - 2m + 20$

$= (m+2)(m^2 - 6m + 10)$

find roots $m = 3 \pm i$ by quadratic formula

$\therefore m_1 = -2$, $m_2 = 3+i$, $m_3 = 3-i$; so the soln to the homog. eqn. is

$$\boxed{y_c = Ae^{-2x} + Be^{3x}\cos x + Ce^{3x}\sin x}$$

As for y_p: we want $L(y_p) = e^{-2x}$.

we know $L(e^{-2x}) = 0$ (from y_p) & $m = -2$ is a simple root of the aux. eqn.. So we need one extra power of x.

conclusion find $L(xe^{-2x})$ to be able to "guess" y_p.

(8.4)

specifically: (20)· $xe^{-2x} = xe^{-2x} = xe^{-2x}$

(−2)· $(xe^{-2x})' = e^{-2x} - 2xe^{-2x} = -2xe^{-2x} + e^{-2x}$

(−4)· $(xe^{-2x})'' = -2e^{-2x} + 4xe^{-2x} - 2e^{-2x} = 4xe^{-2x} - 4e^{-2x}$

(1)· $(xe^{-2x})''' = 4e^{-2x} - 8xe^{-2x} + 8e^{-2x} = -8xe^{-2x} + 12e^{-2x}$

$$L[xe^{-2x}] = \cancel{(20 + 4 - 16 - 8)\, xe^{-2x}} + (-2 + 16 + 12)e^{-2x}$$

$$= 26e^{-2x}$$

so $y_p = \frac{1}{26}xe^{-2x}$ & the general solution is

$$\boxed{y = y_c + y_p = Ae^{-2x} + Be^{3x}\cos x + Ce^{3x}\sin x + \frac{1}{26}xe^{-2x}}$$

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4. for y_c, note that $4x^2y'' + y = 0$ is a Cauchy-Euler equation

with auxiliary eqn $4r(r-1) + 1 = 0 \iff 4r^2 - 4r + 1 = 0 \iff (2r-1)^2 = 0$

so roots $r_1 = r_2 = \frac{1}{2}$, sols $y_1 = x^{\frac{1}{2}}$, $y_2 = x^{\frac{1}{2}}\ln x$

& $\boxed{y_c = Ax^{\frac{1}{2}} + Bx^{\frac{1}{2}}\ln x}$.

For y_p, normalize the DE to get $y'' + \frac{1}{4x^2}y = \frac{4x^{-3/2}}{1+\ln x}$

& use variation of parameters: $y_p = y_1v_1 + y_2v_2 = x^{\frac{1}{2}}v_1 + x^{\frac{1}{2}}\ln x\, v_2$,

where

$$\begin{cases} y_1v_1' + y_2v_2' = 0 \\ y_1'v_1' + y_2'v_2' = \dfrac{4x^{-3/2}}{1+\ln x} \end{cases}$$

$$\Longleftrightarrow \begin{cases} (x^{1/2})u_1' + (x^{1/2}\ln x)u_2' = 0 & \bullet\, x^{-1/2} \quad \textcircled{\scriptsize P.S} \\ \left(\tfrac{1}{2}x^{-1/2}\right)u_1' + \left(\tfrac{1}{2}x^{-1/2}\ln x + x^{-1/2}\right)u_2' = \dfrac{4x^{-3/2}}{1+\ln x} & \bullet\, x^{1/2} \end{cases}$$

$$\Longleftrightarrow \begin{cases} u_1' + (\ln x)u_2' = 0 \\ \tfrac{1}{2}u_1' + \left(\tfrac{1}{2}\ln x + 1\right)u_2' = \dfrac{4x^{-1}}{1+\ln x} \end{cases} \Big) \left(-\tfrac{1}{2}\right)$$

$$\Longleftrightarrow \begin{cases} u_1' + (\ln x)u_2' = 0 \\ \boxed{u_2' = \dfrac{4x^{-1}}{1+\ln x}} \end{cases} \Longrightarrow \boxed{u_1' = \dfrac{(-\ln x)\cdot 4x^{-1}}{1+\ln x}}$$

[N.B. the above is equivalent to solving for u_1' & u_2' by Gaussian elimination].

Then $$\boxed{u_2 = \int \frac{4x^{-1}dx}{1+\ln x} = 4\int \frac{d(1+\ln x)}{1+\ln x} = 4\ln(1+\ln x) \quad \text{(really, } 4\ln|1+\ln x|\text{)}}$$

$$u_1 = -4\int \frac{\ln x \cdot x^{-1}dx}{1+\ln x} = -4\int \frac{w\,dw}{1+w}$$

with $w = \ln x$.
note $\frac{w}{1+w} = 1 - \frac{1}{1+w}$
(e.g. by $\begin{array}{r} 1 \\ 1+w\,\overline{)\,w} \\ \underline{w+1} \\ -1 \end{array}$)

$$= -4\int\left(dw - \frac{dw}{1+w}\right) = -4w + 4\ln(1+w)$$

$$\Rightarrow \boxed{u_1 = -4\ln x + 4\ln(1+\ln x)}$$ (really, we should write $\ln|1+\ln x|$

$$\& \quad y_p = y_1u_1 + y_2u_2 = x^{1/2}\left(-4\ln x + 4\ln(1+\ln x)\right) + x^{1/2}\ln x\left(4\ln(1+\ln x)\right)$$

$$\therefore \boxed{\begin{aligned} y = y_c + y_p = {} & Ax^{1/2} + Bx^{1/2}\ln x \\ & + x^{1/2}\left[-4\ln x + 4\ln(1+\ln x)\right] \\ & + 4x^{1/2}\ln x \cdot \ln(1+\ln x) \end{aligned}}$$

$\boxed{5.}$ $x=0$ is an ordinary point, so we put $y=\sum_{n=0}^{\infty} c_n x^n$.

Then $y'=\sum_{n=0}^{\infty} n c_n x^{n-1}$, $y''=\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, & the DE is

$$y'' + x^2 y'' - 6xy' + 10y = 0$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} 6nc_n x^n + \sum_{n=0}^{\infty} 10c_n x^n}_{\text{here } n(n-1)-6n+10=n^2-7n+10=(n-2)(n-5)}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} (n-2)(n-5)c_n x^n = 0$$

$$\Big\downarrow \begin{array}{l} n-2=k \\ n=k+2 \\ n=0 \Leftrightarrow k=-2 \end{array} \qquad\qquad \Big\downarrow\ n=k$$

$$\sum_{k=-2}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} (k-2)(k-5)c_k x^k = 0$$

Start-up eqns $\left.\begin{array}{l} k=-2 : 0c_0=0 \\ k=-1 : 0c_1=0 \end{array}\right\} \Rightarrow c_0, c_1$ arbitrary

recurrence for $k\geq 0$: $(k+2)(k+1)c_{k+2} + (k-2)(k-5)c_k = 0$

$$\boxed{c_{k+2} = -\frac{(k-2)(k-5)c_k}{(k+2)(k+1)}, \text{ for } k\geq 0}$$

$$c_0 \xrightarrow[{[k=0]}]{\cdot -\frac{(-2)(-5)}{2\cdot 1}} c_2 \xrightarrow[{[k=2]}]{\cdot 0} \underset{\substack{\| \\ 0}}{c_4} \xrightarrow{\cdot \text{any}} \underset{\substack{\| \\ 0}}{c_6} \xrightarrow{\cdot \text{any}} \underset{\substack{\| \\ 0}}{c_8} \to \cdots \to \underset{\substack{\| \\ 0}}{c_{2l}}$$

$\boxed{\text{for } 2l\geq 4}$

$$c_1 \xrightarrow[{[k=1]}]{-\frac{(-1)(-4)}{3\cdot 2}} c_3 \xrightarrow[{[k=3]}]{-\frac{(1)(-2)}{5\cdot 4}} c_5 \xrightarrow[{[k=5]}]{\cdot 0} \underset{\substack{\| \\ 0}}{c_7} \xrightarrow{\text{any}} \underset{\substack{\| \\ 0}}{c_9} \to \cdots \to c_{2l+1}=0$$

$\boxed{\text{for } 2l+1\geq 7}$

$$y = c_0 + c_2x^2 + c_1 + c_3x^3 + c_5x^5 = \boxed{c_0\left(1 - \frac{(-2)(-5)}{2\cdot 1}x^2\right) + c_1\left(x - \frac{(-1)(-4)}{3\cdot 2}x^3 + \frac{(-1)(-4)(1)(-2)}{3\cdot 2\cdot 5\cdot 4}x^5\right)}$$

(everything else is zero)

Note for ex 5 : you can simplify & get

$$y = c_0\left(1-5x^2\right) + c_1\left(x - \frac{2}{3}x^3 - \frac{1}{15}x^5\right)$$

$\boxed{6.}$ if $y = x^m u$, then

$\cdot\left(x^2+\frac{11}{4}\right)$

$\cdot -5x$

$\cdot x^2$

$$y' = mx^{m-1}u + x^m u'$$

$$y'' = m(m-1)x^{m-2}u + 2m x^{m-1}u' + x^m u''$$

$$0 = \left[x^{m+2} + \frac{11}{4}x^m - 5mx^m + m(m-1)x^m\right]u + \left[-5x^{m+1} + 2mx^{m+1}\right]u' + \left[x^{m+2}\right]u'' \quad ⊛$$

since we want our Bessel DE to have the form $x^2u'' + xu' + (x^2-\nu^2)u = 0$, let's FIRST cancel x^m from ⊛ :

$$0 = \left[x^2 + m^2 - 6m + \frac{11}{4}\right]u + \left[(2m-5)x\right]u' + x^2u''$$

this shows us that we want : ① $2m-5=1$

② $m^2 - 6m + \frac{11}{4} = -\nu^2$.

from ①, $\boxed{m=3}$, & substituting into ②, $9 - 18 + \frac{11}{4} = -\nu^2 \Rightarrow \nu^2 = +\frac{25}{4}$

$\Rightarrow \nu = \frac{5}{2}$, NOT an integer.

for ~~this~~ choice of $\overset{m}{\overset{\parallel}{3}}$, we see that $x^2u'' + xu' + \left(x^2 - \left(\frac{5}{2}\right)^2\right)u = 0$

so $u = A J_{5/2}(x) + B J_{-5/2}(x)$

$$\boxed{\& \; y = x^3 u = A x^3 J_{5/2}(x) + B x^3 J_{-5/2}(x)}$$

$\boxed{7.}$ the equation says

$$f - \left(te^{-2t}\right) * f = e^{t} \qquad \text{(check this!)}$$

Taking $\mathcal{L}$, we get:

$$F - \left(\frac{1}{(s+2)^2}\right)F = \frac{1}{s-1}$$

[note: $\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$ from the shifting theorem applied to $\mathcal{L}\{t\} = \frac{1}{s^2}$.
You can also use $\mathcal{L}\{tg(t)\} = -\frac{d}{ds}G(s)$, but this would be uglier.]

after some algebra,

$$F = \frac{(s+2)^2}{(s-1)\left((s+2)^2-1\right)}$$

$$= \frac{(s+2)^2}{(s-1)\underbrace{(s+3)(s+1)}_{\uparrow}} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{C}{s+1}.$$

either from $s^2+4s+4-1$ or by factoring the difference of 2 squares

Find A, B, C by the cover-up method:

$$A = \left.\frac{(s+2)^2}{(s+3)(s+1)}\right|_{s=1} = \frac{9}{8}$$

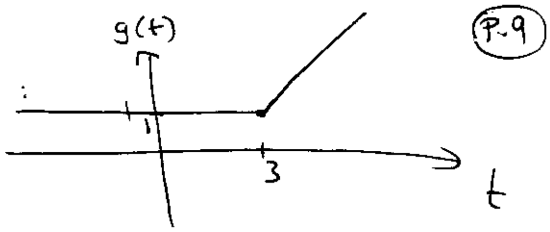
$$B = \left.\frac{(s+2)^2}{(s-1)(s+1)}\right|_{s=-3} = \frac{1}{8}$$

$$C = \left.\frac{(s+2)^2}{(s-1)(s+3)}\right|_{s=-1} = \frac{-1}{4}$$

$$\therefore F = \frac{9/8}{s-1} + \frac{1/8}{s+3} + \frac{-1/4}{s+1}$$

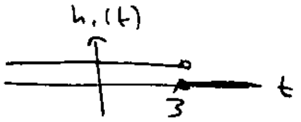
$$\& \quad \boxed{f = \tfrac{9}{8}e^{t} + \tfrac{1}{8}e^{-3t} - \tfrac{1}{4}e^{-t}}$$

$\boxed{8.}$ $g(t) = \begin{cases} 1 & \text{if } t<3 \\ t-2 & \text{if } t\geq 3 \end{cases}$

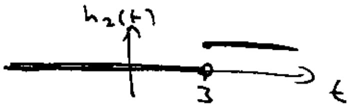


$g(t) = 1 \cdot h_1(t) + (t-2) \cdot h_2(t)$

where $h_1(t) = \begin{cases} 1 & \text{if } t<3 \\ 0 & \text{if } t\geq 3 \end{cases} = 1 - u(t-3)$:



$h_2(t) = \begin{cases} 0 & \text{if } t<3 \\ 1 & \text{if } t\geq 3 \end{cases} = u(t-3)$:



So $g(t) = 1 - u(t-3) + (t-2)u(t-3) = 1 + (t-3)u(t-3)$

& $\boxed{G(s) = \frac{1}{s} + \frac{e^{-3s}}{s^2}}$ (the latter from $\mathcal{L}\{f(t-3)u(t-3)\} = e^{-3s}F(s)$
here $f(t)=t$, so $F = \frac{1}{s^2}$)

so the translation of our DE into Laplace world becomes:

$$s(sY - y(0)) - y'(0) + 9Y = G(s)$$

$$s^2Y - s - 0 + 9Y = \frac{1}{s} + \frac{e^{-3s}}{s^2}$$. with some algebra, we get

$$\boxed{Y = \frac{s^2+1}{s(s^2+9)} + \frac{e^{-3s}}{s^2(s^2+9)}}$$

Remarks: it is usually best to combine fractions of the form $\frac{(\text{polynomial})}{(\text{polynomial})}$

BUT also make sure to keep each factor of an exponential e^{-as} separate!

so the form should be $Y = \frac{\text{poly}(s)}{\text{poly}(s)} + e^{-as}\frac{\text{poly}(s)}{\text{poly}(s)}$
$+ e^{-bs} \cdot \frac{\text{poly}(s)}{\text{poly}(s)}$, for example.

now do partial fractions:

$$\frac{s^2+1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$, where $A = \frac{1}{9}$ (cover-up)
& $B = 8/9$, $C = 0$ (direct solution of $s^2+1 = A(s^2+9) + (Bs+C)s$)

$$= \frac{1/9}{s} + \frac{8/9\, s}{s^2+9}$$

and $$\frac{1}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$\underbrace{= \ldots}_{\text{check it yourself}} = \frac{1/9}{s^2} + \frac{-1/9}{s^2+9}$$

(i.e. $A = C = 0$, $\underbrace{B = 1/9}_{\text{e.g. by cover-up}}$, $D = -1/9$).

Conclusion $$Y = \frac{1/9}{s} + 8/9\,\frac{s}{s^2+9} + e^{-3s}\left(\frac{1/9}{s^2} - \frac{1}{27}\cdot\frac{3}{s^2+9}\right)$$

$$\boxed{y = 1/9 + 8/9 \cos 3t + u(t-3)\left[\frac{1}{9}(t-3) - \frac{1}{27}\sin 3(t-3)\right]}.$$

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9. the characteristic eqn is $\det(A - \lambda I) = 0$

i.e. $$\det\begin{pmatrix} 2-\lambda & 4 & 0 \\ 0 & -1-\lambda & -5 \\ 0 & 1 & 1-\lambda \end{pmatrix} = 0$$

best: expand by 1st column

get $$(2-\lambda)\cdot\det\begin{pmatrix} -1-\lambda & -5 \\ 1 & 1-\lambda \end{pmatrix} - 0 + 0 = 0$$

$$(2-\lambda)\cdot\left[(-1-\lambda)(1-\lambda) + 5\right] = 0$$

$$(2-\lambda)(\lambda^2+4) = 0$$

so $$\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 2i \\ \lambda_3 = -2i \end{cases}$$

3 distinct (complex) roots, no problem with multiple roots.

$\underline{\lambda_1 = 2}$: $(A - 2I)\vec{K}_1 = \vec{0}$

must solve $\left(\begin{array}{ccc|c} 0 & 4 & 0 & \\ 0 & -3 & -5 & \vec{0} \\ 0 & 1 & -1 & \end{array}\right) \cdot \frac{1}{4}$

$\Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & \vec{0} \\ 0 & -3 & -5 & \\ 0 & 1 & -1 & \end{array}\right) \begin{array}{l} +3 \\ -1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & \vec{0} \\ 0 & 0 & -5 & \\ 0 & 0 & -1 & \end{array}\right) \cdot(-1) \text{ \& move up}$

$\Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ 0 & 0 & 1 & \vec{0} \\ 0 & 0 & -5 & \end{array}\right) +5 \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & \vec{0} \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{array}\right)$

i.e $k_2 = 0$, $k_3 = 0$ $\qquad k_2, k_3$ <u>bound</u> (both zero!) $\quad k_1$ <u>free</u>

$\vec{K}_1 = \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. we can take $k_1 = 1$ so $\vec{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

& $\boxed{\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$.

$\underline{\lambda_2 = 2i}$: $(A - 2iI)\vec{K}_2 = \vec{0}$

must solve $\left(\begin{array}{ccc|c} 2-2i & 4 & 0 & \\ 0 & -1-2i & -5 & \vec{0} \\ 0 & 1 & 1-2i & \end{array}\right) \underbrace{\cdot \frac{1}{2-2i} \cdot \frac{2+2i}{2+2i}}_{\text{i.e } \cdot \frac{(1+i)}{4}}$

also exchange

(note the "2" cancels)

$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1+i & 0 & \\ 0 & 1 & 1-2i & \vec{0} \\ 0 & -1-2i & -5 & \end{array}\right) \begin{array}{l} -(1+i) \\ +(1+2i) \end{array}$

$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3+i & \\ 0 & 1 & 1-2i & \vec{0} \\ 0 & 0 & 0 & \end{array}\right)$

here, $0 - (1+i)(1-2i)$
$= -1 + 2i^2 - i + 2i$
$= -1 - 2 - i + 2i$
$= -3 + i$

here, $-5 + (1+2i)(1-2i)$
$= -5 + 1 - 4i^2 - 2i + 2i = -5 + 1 + 4$
$= 0$

so $$\begin{cases} k_1 \qquad + (-3+i)\,k_3 = 0 \\ \qquad k_2 + (1-2i)\,k_3 = 0 \end{cases}$$

k_1, k_2 bound : $$\left.\begin{aligned} k_1 &= (3-i)\,k_3 \\ k_2 &= (-1+2i)\,k_3 \end{aligned}\right\} \Rightarrow \vec{K_2} = k_3 \begin{pmatrix} 3-i \\ -1+2i \\ 1 \end{pmatrix}$$

k_3 free .

we can choose $k_3 = 1$ so $\vec{K_2} = \begin{pmatrix} 3-i \\ -1+2i \\ 1 \end{pmatrix}$,

$$\boxed{\vec{X_2} = e^{2it} \begin{pmatrix} 3-i \\ -1+2i \\ 1 \end{pmatrix}} .$$

$\underline{\lambda_3 = -2i}$: get $\vec{K_3} = \begin{pmatrix} 3+i \\ -1-2i \\ 1 \end{pmatrix}$ (complex conjugate of $\vec{K_2}$)

& $\vec{X_3} = e^{-2it} \begin{pmatrix} 3-i \\ -1+2i \\ 1 \end{pmatrix}$

so <u>complex solution</u> $\vec{X} = A\vec{x_1} + B\vec{x_2} + C\vec{x_3}$

$$\boxed{\vec{X} = A e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + B e^{2it} \begin{pmatrix} 3-i \\ -1+2i \\ 1 \end{pmatrix} + C e^{-2it} \begin{pmatrix} 3+i \\ -1-2i \\ 1 \end{pmatrix}}$$

(if you want the <u>real solution</u>, it is

$$\vec{X} = A e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + D \left[\cos 2t \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right] + E \left[\cos 2t \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right]$$

; <u>check!</u>)

$\boxed{10.}$ the characteristic equation is

$$\det\begin{pmatrix}-\lambda & -9\\ 1 & 6-\lambda\end{pmatrix} = 0$$

$\iff -\lambda(6-\lambda)+9=0 \iff \lambda^2-6\lambda+9=0 \iff (\lambda-3)^2=0$

double eigenvalue $\boxed{\lambda_1=\lambda_2=3}$.

let's see if we are in the lucky case with two independent eigenvectors:

$$(A-3I)\vec{K}=\vec{0} \iff \left(\begin{array}{cc|c}-3 & -9 & \vec{0}\\ 1 & +3 & \end{array}\right) \underset{\text{easy}}{\iff} \left(\begin{array}{cc|c}1 & +3 & \vec{0}\\ 0 & 0 & \end{array}\right)$$

so $k_1+3k_2=0$, $\vec{K}=\begin{pmatrix}-3k_2\\ k_2\end{pmatrix}=k_2\begin{pmatrix}-3\\ 1\end{pmatrix}$ general soln for $\vec{K}$,

we only get one indep. eigenvector $\vec{K}=\begin{pmatrix}-3\\ 1\end{pmatrix}$ & $\boxed{\overset{\boxed{\text{ONE}}}{\vec{X}_1}=e^{3t}\begin{pmatrix}-3\\ 1\end{pmatrix}}$.

We need a second soln of the form $\vec{X}_2 = te^{3t}\vec{L}+e^{3t}\vec{M}$

where ① $(A-3I)\vec{L}=\vec{0}$, same eqn as for $\vec{K}$, so we can take $\vec{L}=\vec{K}=\begin{pmatrix}-3\\ 1\end{pmatrix}$

② $(A-3I)\vec{M}=\vec{L}$, $\iff \left(\begin{array}{cc|c}-3 & -9 & -3\\ 1 & 3 & 1\end{array}\right) \underset{\text{easy}}{\iff} \left(\begin{array}{cc|c}1 & 3 & 1\\ 0 & 0 & 0\end{array}\right)$

$\iff m_1+3m_2=1$, $\vec{M}=\begin{pmatrix}1-3m_2\\ m_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}+m_2\begin{pmatrix}-3\\ 1\end{pmatrix}$

we only need one $\vec{M}$, so let's take $m_2=0$, $\vec{M}=\begin{pmatrix}1\\ 0\end{pmatrix}$, (other choices are possible)

so $\boxed{\vec{X}_2 = te^{3t}\begin{pmatrix}-3\\ 1\end{pmatrix}+e^{3t}\begin{pmatrix}1\\ 0\end{pmatrix}}$ 2nd solution

$$\boxed{\text{general soln } \vec{X}=A\vec{X}_1+B\vec{X}_2 = Ae^{3t}\begin{pmatrix}-3\\ 1\end{pmatrix}+B\left(te^{3t}\begin{pmatrix}-3\\ 1\end{pmatrix}+e^{3t}\begin{pmatrix}1\\ 0\end{pmatrix}\right)}$$