MATHEMATICS 202
SECOND SEMESTER, 2007-08
Solutions of Quiz I

Time: 70 Minutes

Date: MARCH 15, 2008

Name:——————————-

ID Number:————————

Section:——————————-

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Instructions:
(a) Write with a pen; never use a pencil.
(b) Answers must be fully justified.
(c) The grade allocated to each question is set next to it.
(d)**Answer The Following Seven Questions On The Page Allocated For Each Question (You May Use The Back Of The Pages If Needed).**

Question	Grade
1	/10
2	/10
3	/10
4	/15
5	/15
6	/15
TOTAL	/75

(1) Use Green's theorem to evaluate the line integral

$$\oint_C y^2\, dx + x^2\, dy,$$

where C is the positively-oriented boundary of the region bounded by the semi-circle $y = \sqrt{4-x^2}$ and the $x-$axis. (10 points)

Solution. If R is the region bounded by C, then by Green's theorem we have:

$$\begin{aligned}
\oint_C y^2\, dx + x^2\, dy &= \int\int_R \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right) dA \\
&= 2\int\int_R (x-y)dA \\
&= 2\int_0^\pi \int_0^2 (\cos\theta - \sin\theta)r^2 dr d\theta \\
&= 2\int_0^\pi (\cos\theta - \sin\theta)\left[r^3/3\right]_0^2 d\theta \\
&= \frac{16}{3}\int_0^\pi (\cos\theta - \sin\theta)d\theta \\
&= \frac{16}{3}\left[\sin\theta + \cos\theta\right]_0^\pi \\
&= \frac{-32}{3}
\end{aligned}$$

(2) Use the Divergence theorem to find the outward flux of the vector field

$$\mathbf{F} = x^3\mathrm{i} + y^3\mathrm{j} + z^3\mathrm{k}$$

across the surface of the solid that lies inside the sphere $x^2+y^2+z^2 = 1$ and above the $xy-$plane. (10 points)

Solution. If R is the region bounded by the surface S of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 1$ and above the $xy-$plane, then the desired flux is

$$\begin{aligned}
\int\int_S \mathrm{F.n}\, d\sigma &= \int\int\int_R \nabla.\mathrm{F}\, dV \\
&= \int\int\int_R \left(\frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3\right) dV \\
&= 3\int\int\int_R (x^2 + y^2 + z^2) dV \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4 \sin\phi\; d\rho d\phi d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2} [\rho^5/5]_0^1 \sin\phi\; d\phi d\theta \\
&= \frac{3}{5}\int_0^{2\pi} [-\cos\phi]_0^{\pi/2} d\theta \\
&= \frac{6\pi}{5}
\end{aligned}$$

(3) Use Stoke's theorem to find the circulation of the vector field

$$\mathbf{F} = x^2\mathrm{i} + y^2\mathrm{j} + z^2\mathrm{k}$$

on the common circle C of the paraboloid $z = 4 - x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$. (10 points)

Solution. By Stoke's theorem, if S is the cap of the paraboloid cut from the paraboloid by the plane $z = 3$, then the desired circulation is

$$\oint_C \mathbf{F}.\mathbf{T}\, ds = \int\int_S \nabla \times \mathbf{F}.\mathbf{n}\, d\sigma = 0$$

since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

(4) Consider the force vector field

$$\mathbf{F} = (yz - x)\mathrm{i} + (xz - y)\mathrm{j} + (xy - z)\mathrm{k}.$$

(a) Show that $\mathbf{F}$ is conservative. (5 points)

Solution. The field $\mathbf{F}$ is conservative since

$$\frac{\partial}{\partial y}(yz - x) = z = \frac{\partial}{\partial x}(xz - y),$$
$$\frac{\partial}{\partial z}(xz - y) = x = \frac{\partial}{\partial y}(xy - z),$$
$$\frac{\partial}{\partial z}(yz - x) = y = \frac{\partial}{\partial x}(xy - z).$$

(b) Find a potential function f of $\mathbf{F}$. (5 points)

Solution. If f is the potential function of $\mathbf{F}$, then

$$f_x = yz - x, \quad f_y = xz - y, \quad f_z = xy - z.$$

Consequently,

$$f(x, y, z) = \int f_x \, dx + g(y, z) = \int (yz - x) \, dx + g(y, z) = xyz - x^2/2 + g(y, z).$$

Hence,

$$f_y = xz + \frac{\partial g}{\partial y} = xz - y,$$

and

$$\frac{\partial g}{\partial y} = -y, \ \text{ or } \ g(y, z) = -y^2/2 + h(z).$$

Thus $f(x, y, z) = xyz - x^2/2 - y^2/2 + h(z)$; hence

$$f_z = xy + h'(z) = xy - z, \ \text{ or } \ h'(z) = -z, \ \text{ or } \ h(z) = -z^2/2.$$

Thus a potential function for $\mathbf{F}$ is

$$f(x, y, z) = xyz - x^2/2 - y^2/2 - z^2/2.$$

(c) Find the work done by $\mathbf{F}$ on the curve C that runs from $(1, 3, 2)$ to $(2, 1, -3)$. (5 points)

Solution. Since a potential function for $\mathbf{F}$ is $f(x, y, z) = xyz - x^2/2 - y^2/2 - z^2/2$, the work done by $\mathbf{F}$ on the curve C that runs from $(1, 3, 2)$ to $(2, 1, -3)$ is

$$\oint_C \mathbf{F}.\mathbf{T} \, ds = f(2, 1, -3) - f(1, 3, 2) = -12.$$

(5) Consider the surface of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 4$.

(a) Find a parametrization for the surface in terms of cylindrical coordinates. (5 points)

Solution. A parametrization for the surface in terms of cylindrical coordinates is

$$\mathbf{r}(r, \theta) = (r \cos\theta)\,\mathbf{i} + (r \sin\theta)\,\mathbf{j} + r\,\mathbf{k}, \quad 0 \le r \le 4,\ 0 \le \theta \le 2\pi.$$

(b) Use the parametrization to find the surface area. (10 points)

Solution. Since

$$\mathbf{r}_r(r, \theta) = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j} + \ \mathbf{k}$$

and

$$\mathbf{r}_\theta(r, \theta) = (-r \sin\theta)\,\mathbf{i} + (r \cos\theta)\,\mathbf{j},$$

we have

$$\begin{aligned} |\mathbf{r}_r \times \mathbf{r}_\theta| &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\ &= |(-r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}| \\ &= \sqrt{2}r, \end{aligned}$$

and $d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta| dr\ d\theta = \sqrt{2}r\ dr\ d\theta$. Thus the desired area of the surface S is

$$\begin{aligned} A = \int\int_S d\sigma &= \int_0^{2\pi} \int_0^4 \sqrt{2}r\ dr\ d\theta \\ &= \sqrt{2}(\int_0^{2\pi} d\theta)(\int_0^4 r\ dr) \\ &= \sqrt{2}(2\pi)[r^2/2]_0^4 = 16\sqrt{2}\pi. \end{aligned}$$

(6) Consider the initial-value problem: $dy/dx = 8x^3\sqrt{y}, \quad y(0) = 0$.

(a) Show that the functions $y_1(x) = x^8, \ -\infty < x < \infty$, and

$$y_2(x) = \begin{cases} 0, & x < 0 \\ x^8, & x \geq 0 \end{cases}$$

are solutions of the initial-value problem. (5 points)

Solution. Since $y_1'(x) = 8x^7 = 8x^3\sqrt{x^8} = 8x^3\sqrt{y}$ for all $x, \ -\infty < x < \infty$,

$$y_2'(x) = \begin{cases} 0, & x < 0 \\ 8x^7, & x \geq 0 \end{cases},$$

$y_2'(x) = 8x^7 = 8x^3\sqrt{y_2} = 8x^3\sqrt{y_2}$ for all $x, \ -\infty < x < \infty$, and $y_1(0) = 0$ $y_2(0) = 0$, y_1 and y_2 are solutions for the IVP.

(b) Does the initial-value problem satisfy the existence and uniqueness theorem for first-order differential equations on $]-\infty, \infty[$? Justify your answer carefully. (5 points)

Solution. NO since

$$\frac{\partial}{\partial y} 8x^3\sqrt{y} = \frac{4x^3}{\sqrt{y}}$$

is not continuous at the origin.

(c) Point out the apparent contradiction between the the results of (a) and (b) and resolve the contradiction. Justify your answer carefully. (5 points)

Solution. The apparent contradiction is that the IVP has two solutions whereas the initial-value problem satisfy the existence and uniqueness theorem for first-order differential equations on suggests having a unique solution. But the contradiction is not valid as the conditions of the existence and uniqueness theorem, namely the continuity of $\partial f/\partial y$, with $f(x, y) = 8x^3\sqrt{y}$, does not hold at the origin.