MATHEMATICS 202
SPRING SEMESTER 2006-2007
QUIZ I

Time: 80 MINUTES.

Date: March 26, 2007.

Name:————————————————————-

ID Number:————————————————————

Section Number:————————————————————

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Question	Grade
1	/14
2	/14
3	/14
4	/14
5	/14
6	/14
7	/18
TOTAL	/100

Answer The Following Seven Questions On The Page Allocated For Each Question (You May Use The Back Of The Pages If Needed).

1. Show that the differential equation

$$(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0$$

is not exact. Multiply the equation by an appropriate integrating factor $\mu(x,y) = x^m y^n$ that makes the differential equation exact, then solve.

(14 points)

Solution. Since

$$(-xy\sin x+2y\cos x)_y = -x\sin x+2\cos x \neq 2\cos x-2x\sin x = (2x\cos x)_x,$$

the differential equation is not exact.

Multiply both sides of the differential equation by $\mu(x,y) = x^m y^n$ to obtain

$$(-x^{m+1}y^{n+1}\sin x + 2x^m y^{n+1}\cos x)\,dx + 2x^{m+1}y^n\cos x\,dy = 0.$$

For this equation to be exact we must have

$$(-x^{m+1}y^{n+1}\sin x + 2x^m y^{n+1}\cos x)_y = (2x^{m+1}y^n\cos x)_x$$

or

$$-(n+1)x^{m+1}y^n\sin x+2(n+1)x^m y^n\cos x = 2(m+1)x^m y^n\cos x-2x^{m+1}y^n\sin x.$$

This implies that $m = n = 1$. Hence the differential equation

$$(-x^2y^2\sin x + 2xy^2\cos x)\,dx + 2x^2y\cos x\,dy = 0$$

is exact. Let

$$f_x = -x^2y^2\sin x + 2xy^2\cos x \quad \text{and} \quad f_y = 2x^2y\cos x.$$

Then

$$f(x,y) = \int 2x^2y\cos x\,dy = x^2y^2\cos x + g(x).$$

Hence,

$$2xy^2\cos x - x^2y^2\sin x + g'(x) = f_x = -x^2y^2\sin x + 2xy^2\cos x$$

which yields $g'(x) = 0$. Hence, $f(x,y) = x^2y^2\cos x$ and the general solution is

$$x^2y^2\cos x = c$$

for any constant c.

2. Solve the initial-value problem (14 points)

$$x^2\frac{dy}{dx} - 2xy = 3y^4, \qquad y(1) = 1/2.$$

Solution. Write the differential equation as

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4.$$

This it is a Bernoulli's differential equation. Then write the differential equation as

$$y^{-4}\frac{dy}{dx} - \frac{2}{x}y^{-3} = \frac{3}{x^2}.$$

Letting $u = y^{-3}$ yields

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \quad \text{or} \quad y^{-4}\frac{dy}{dx} = \frac{-1}{3}\frac{du}{dx}.$$

Then substitution in the latter differential equation yields

$$\frac{-1}{3}\frac{du}{dx} - \frac{2}{x}u = \frac{3}{x^2},$$

or

$$\frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2}.$$

An integrating factor for this equation is

$$\mu(x) = e^{\int (6/x)dx} = x^6.$$

Multiplying the latter differential equation by μ yields

$$x^6\frac{du}{dx} + 6x^5u = -9x^4 \quad \text{or} \quad \frac{d}{dx}(x^6u) = -9x^4.$$

Now integration of both sides gives

$$x^6u = -\frac{9}{5}x^5 + c \quad \text{or} \quad y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}.$$

But $y(1) = 1/2$ gives $8 = -9/5 + c$, or $c = 9/5$. Therefore, the solution for the IVP is

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}.$$

3. Solve the initial-value problem (14 points)

$$\left(\frac{y}{x} - \frac{x^2}{y^2}\right)\, dx - \; dy = 0, \qquad y(1) = 2.$$

Solution. Write the differential equation as

$$(y^3 - x^3)dx - xy^2\, dy = 0.$$

Thus this is a homogeneous differential equation. If

$$y = ux, \quad \text{then} \quad dy = u\, dx + x\, du.$$

Then substitution in the latter differential equation yields

$$x^3(u^3 - 1)dx - x^3u^2(u\, dx + x\, du) = 0,$$

or

$$(u^3 - 1)dx - u^2(u\, dx + x\, du) = 0,$$

or

$$dx + xu^2\, du = 0.$$

Write this differential equation as

$$u^2\, du + \frac{dx}{x} = 0,$$

then integrate both sides to obtain

$$\frac{1}{3}u^3 + \ln|x| = c, \quad \text{or} \quad \frac{y^3}{3x^3} + \ln|x| = c.$$

But $y(1) = 2$ yields $c = 8/3$. Therefore, the solution for the IVP is

$$y^3 + 3x^3 \ln|x| = 8x^3.$$

4. Solve the initial-value problem (14 points)

$$\frac{dy}{dx} = e^{x+y} + e^{-x-y} - 1, \qquad y(0) = 0.$$

Determine the largest interval of definition of the solution.

Solution. If $u = x + y$, then

$$\frac{du}{dx} = 1 + \frac{dy}{dx}, \quad \text{or} \quad \frac{dy}{dx} = \frac{du}{dx} - 1.$$

Then substitution in the differential equation yields

$$\frac{du}{dx} - 1 = e^u + e^{-u} - 1, \quad \text{or} \quad \frac{du}{dx} = e^u + e^{-u}.$$

Write this differential equation as

$$\frac{e^u}{e^{2u} + 1} du = dx,$$

then integrate both sides to obtain

$$\arctan e^u = x + c, \quad \text{or} \quad e^{x+y} = \tan(x + c).$$

But $y(0) = 0$ yields $c = \pi/4$. Therefore, the solution for the IVP is

$$e^{x+y} = \tan(x + \pi/4), \quad \text{or} \quad y = \ln(\tan(x + \pi/4)) - x.$$

5. Sketch the largest region in the $xy-$plane containing the points (x_0, y_0) for which the initial-value problem

$$\sqrt{1-y^2}\,\frac{dy}{dx} = \sqrt{1+x^2}, \qquad y(x_0) = y_0,$$

possesses a unique solution; justify your answer. Justify your answer.

(14 points)

Solution. Write this differential equation as

$$\frac{dy}{dx} = \frac{\sqrt{1+x^2}}{\sqrt{1-y^2}},$$

and let

$$f(x,y) = \frac{\sqrt{1+x^2}}{\sqrt{1-y^2}}.$$

Since f and

$$f_y = \frac{y\sqrt{1+x^2}}{(1-y^2)^{3/2}}$$

are continuous functions in the region $R = \{(x,y) : |y| < 1\}$, by the existence and uniqueness theorem, every initial-value problem

$$\sqrt{1-y^2}\,\frac{dy}{dx} = \sqrt{1+x^2}, \qquad y(x_0) = y_0,$$

where $(x_0, y_0) \in R$ possesses a unique solution. Moreover, since f is not defined whenever $y = \pm 1$, R is largest desired region.

6. Use Stoke's theorem to evaluate the circulation of the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x^2\mathbf{j} + z^3\mathbf{k}$$

along the curve C which is the intersection of the circular cylinder $x^2 + y^2 = 4$ and the plane $x+z = 3$, and which is traversed counterclockwise when viewed from high up on the positive $z-$axis. (14 points)

Solution. By Stoke's theorem to evaluate the circulation of the given vector field is given by

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where R is the disc $x^2 + y^2 = 4$, $\mathbf{n}$ and $d\sigma$ are the unit surface normal and the surface area differential for the given surface. Thus,

$$\mathbf{n} = \frac{\nabla(x + z - 3)}{|\nabla(x + z - 3)|} = \frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}$$

and

$$d\sigma = \frac{|\nabla(x + z - 3)|}{|\nabla(x + z - 3) \cdot \mathbf{k}|} dA = \sqrt{2}\, dA.$$

But

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x^2 & z^3 \end{vmatrix} = (2x + 1)\mathbf{k}$$

Therefore, the desired circulation is

$$\begin{aligned} \int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int\int_R (2x + 1)\mathbf{k} \cdot \frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}} \sqrt{2}\, dA \\ &= \int\int_R (2x + 1) dA \\ &= \int_0^{2\pi} \int_0^2 (2r \cos\theta + 1) r\, dr\, d\theta \\ &= \int_0^{2\pi} [2r^3 \cos\theta/3 + r^2/2]_0^2 d\theta = \int_0^{2\pi} [16 \cos\theta/3 + 2] d\theta \\ &= [16 \sin\theta/3 + 2\theta]_0^{2\pi} = 4\pi. \end{aligned}$$

7. Evaluate **in two different ways** the flux of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across the surface S which is the portion of the paraboloid $z = 4-x^2-y^2$ above the $xy-$plane ($\mathbf{n}$ is upward-oriented so that its $z-$component is positive). (16 points)

Solution. By taking $z = 0$ in the equation of the paraboloid, we conclude that the paraboloid meets the $xy-$plane in the circle $x^2+y^2 = 4$. The desired flux is given by $\int\int_S \mathbf{F}\cdot\mathbf{n}\ d\sigma$, where $d\sigma$ is the surface area differential for the paraboloid and

$$\mathbf{n} = \frac{\nabla(x^2+y^2+z-4)}{|\nabla(x^2+y^2+z-4)|} = \frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}.$$

Thus the flux is

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \int\int_S \frac{x^2+y^2+4}{\sqrt{4x^2+4y^2+1}}\ d\sigma.$$

Now we find $d\sigma$ in two ways:

(a)

$$d\sigma = \frac{|\nabla(x^2+y^2+z-4)|}{|\nabla(x^2+y^2+z-4)\cdot\mathbf{k}|}\ dA = (4x^2+4y^2+1)\ dA.$$

In this case, the flux is

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\mathbf{n}\ d\sigma &= \int\int_R (x^2+y^2+4)\ dA \\ &= \int_0^{2\pi}\int_0^2 (r^2+4)r\ dr\ d\theta \\ &= 16\pi.\end{aligned}$$

(b) $d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta$, where

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4-r^2)\mathbf{k}.$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix},$$

and

$$\mathbf{r}_r \times \mathbf{r}_\theta = (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} + r\mathbf{k}.$$

Thus, $d\sigma = \sqrt{4r^4+r^2}\ dr\ d\theta = \sqrt{4r^2+1}\ rdr\ d\theta$. The rest of the work goes as in (a).