MATHEMATICS 202
SPRING SEMESTER 2006-07
QUIZ II

Time: 70 MINUTES.

Date: April 28, 2007.

Name:____-

ID Number:____

Section Number:_____________________________________

Course Instructors: Professors Abdallah Lyzzaik and Hassan Yamani

Question	Grade
1	/20
2	/20
3	/20
4	/20
5	/10
6	/10
TOTAL	/100

Answer The Following Six Questions On The Page Allocated For Each Question (You May Use The Back Of The Pages If Needed).

1. Find the general solution of the differential equation

$$y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$$

knowing that one of its solutions is $y = \cos x$. (20 points)

Solution. Since $\cos x$ is a solution, two characteristic root are $\pm i$. Thus, by Synthetic Division by $\pm i$, the CE is found to be

$$m^4 + 2m^3 + 11m^2 + 2m + 10 = (m - i)(m + i)(m^2 + 2m + 10) = 0$$

and the characteristic root are $\pm i$. and $-1 \pm 3i$. Hence, the GS for the differential equation is

$$y - c_1 \cos x + c_2 \sin x + e^{-x}(c_3 \cos 3x + c_4 \sin 3x), \quad x \in \mathbb{R}.$$

2. Find the general solution of the differential equation

$$x^2y'' - xy' + y = x^2.$$

(20 points)

Solution. This is a nonhomogeneous Cauchy-Euler differential equation with CE

$$m^2 - 2m + 1 = (m-1)^2 = 0.$$

Hence the complementary solution is

$$y_c = c_1 x + c_2 x \ln x.$$

By the method of VP, a particular solution is

$$y_p = u_1 x + u_2 x \ln x,$$

where

$$u_1' = W_1/W \quad \text{and} \quad u_2' = W_2/W,$$

with

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x,$$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = -x\ln x,$$

and

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix} = x.$$

Thus, $u_1' = -\ln x$ and $u_2' = 1$, and consequently, $u_1 = -\int \ln x \, dx = x - x\ln x$ and $u_2 = \int dx = x$. Hence,

$$y_p = (x - x\ln x)x + x^2 \ln x = x^2$$

and the GS is

$$y = c_1 x + c_2 x \ln x + x^2.$$

3. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' + y = e^x + \sin x.$$

(20 points)

Solution. This is a nonhomogeneous linear differential equation with CE

$$m^2 + 1 = 0.$$

Hence the characteristic roots are $m = \pm i$, and the complementary solution is

$$y_c = c_1 \cos x + c_2 \sin x, \quad x \in \mathbb{R}.$$

A particular solution of the differential equation is $y_p = y_{p_1} + y_{p_2}$, where y_{p_1} and y_{p_2} are particular solutions of the differential equation $y'' + y = e^x$ and $y'' + y = \sin x$, respectively.

A particular solution of the first differential equation is given by $y_{p_1} = x(a \cos x + b \sin x)$. Then

$$y'_{p_1} = (a \cos x + bsinx) + x(-a \sin x + b \cos x)$$

and

$$y''_{p_1} = -2a \sin x + 2b \cos x + x(-a \cos x - b \sin x).$$

Hence, by substitution, in the first differential equation, we obtain

$$-2a \sin x + 2b \cos x = \sin x,$$

which gives $a = -1/2$ and $b = 0$. Thus $y_{p_1} = -(x \cos x)/2$.

A particular solution of the second differential equation is given by $y_{p_2} = ae^x$. Then $y'_{p_2} = y''p_2 = ae^x$. Hence, by substitution, in the second differential equation, we obtain $2ae^x = e^x$, or $a = 1/2$. Thus $y_{p_2} = e^x/2$.

Therefore, $y_p = -(x \cos x)/2 + e^x/2$, and the GS is

$$c_1 \cos x + c_2 \sin x - (x \cos x)/2 + e^x/2, \quad x \in \mathbb{R}.$$

4. Use the substitution $x = e^t$ to transform the Cauchy-Euler differential equation

$$x^2y'' + 10xy' + 8y = x^2$$

to a differential equation of constant coefficients, then solve the differential equation. **Show all the details of your work**.

(20 points)

Solution. By substitution, we find

$$y' = \frac{1}{x}\dot{y}, \qquad y'' = \frac{1}{x^2}(\ddot{y} - \dot{y}),$$

and the differential equation becomes

$$\ddot{y} + 9\dot{y} + 8y = e^{2t}.$$

Hence, the CE is $m^2 + 9m + 8 = (m+1)(m+8) = 0$ and the CRs are $-1, -8$. Thus complementary solution of the differential equation is

$$y_c = c_1e^{-t} + c_2e^{-8t}.$$

Let $y_p = Ae^{2t}$. By substitution in the latter differential equation yields,

$$(4A + 18A + 8A)e^{2t} = e^{2t}.$$

Then $A = 1/30$, and $y_p = e^{2t}/30$. therefore, the GS in t is

$$y = c_1e^{-t} + c_2e^{-8t} + e^{2t}/30,$$

and in terms in it is

$$y = c_1x^{-1} + c_2x^{-8} + x^2/30,$$

5. Find the general solution of the differential equation

$$xy'' - (x+1)y' + y = 0$$

knowing that $y_1 = e^x$ is a solution. (10 points)

Solution. Write the DE as

$$y'' - \frac{x+1}{x}y' + \frac{1}{x}y = 0.$$

By the Reduction of Order method,

$$y_2 = e^x \int \frac{\exp \int ((x+1)/x)\, dx}{e^{2x}} dx = e^x \int xe^{-x}\, dx = -x - 1.$$

Hence, the GS is

$$y = c_1 e^x + c_2(x+1), \quad x \in \mathbb{R}.$$

6. Can the set $\{x^2, x^3\}$ be a fundamental set over $(-\infty, \infty)$ for a linear homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

where a_2, a_1, and a_0 are continuous functions of $(-\infty, \infty)$ with $a_2(x) \neq 0$ for all x? **Justify your answer**.

Solution. The set $\{x^2, x^3\}$ is not a fundamental set over $(-\infty, \infty)$ for any of the suggested differential equations since the Wronskian

$$W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$$

is zero when $x = 0$ and nonzero otherwise.

(10 points)