MATHEMATICS 202
SPRING SEMESTER 2007-08
SOLUTION OF QUIZ II

Time: 70 MINUTES

Date: April 19, 2008

Name:——————————————————-

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Course Instructors: Professors Abdallah Lyzzaik and Dolly Fayyad

Question	Grade
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/10
TOTAL	/100

Answer The Following Six Questions On The Page Allocated For Each Question (You May Use The Back Of The Pages If Needed).

1. Solve the initial-value problem

$$x\frac{dy}{dx} + 4y = x^4y^2; \quad y(1) = 1.$$

(15 points)

Solution. By writing the DE as

$$\frac{dy}{dx} + \frac{4}{x}y = x^3y^2,$$

it becomes a Bernoulli DE of degree 2. Now write the latter equation as

$$y^{-2}\frac{dy}{dx} + \frac{4}{x}y^{-1} = x^3,$$

and let $u = y^{-1}$. Then $-u' = y^{-2}y'$ and substitution yields

$$\frac{du}{dx} + \frac{-4}{x}u = -x^3,$$

which is a linear DE of order 1 with an integrating factor

$$\mu(x) = \int \exp\left(\frac{-4}{x}\right)dx = \frac{1}{x^4}.$$

Multiplying μ with the latter DE yields

$$\frac{1}{x^4}\frac{du}{dx} + \frac{-4}{x^5}u = -x^{-1},$$

or

$$\frac{d}{dx}\left(\frac{u}{x^4}\right) = -x^{-1}.$$

Integration of both sides with respect to x gives

$$\left(\frac{u}{x^4}\right) = -\ln|x| + c,$$

or

$$\left(\frac{1}{yx^4}\right) = -\ln|x| + c.$$

Since $y(1) = 1$, $c = 1$ and the solution of the initial-value problem is

$$y = \frac{1}{x^4(1 - \ln|x|)}.$$

2. Find the general solution of the differential equation

$$(3x^2y + y^3)\, dx + (x^3 + 3xy^2)\, dy = 0.$$

(15 points)

Solution. Since $3x^2y+y^3$ and $3x^2y+y^3$ are homogenous functions of same degree 3, the DE is homogenous. If $y = ux$, then $dy = udx + xdu$ and, by substitution the DE becomes

$$x^3(u^2 + 3)dx + x^3(3u^2 + 1)(udx + xdu) = 0,$$

or

$$4u(u^2 + 1)dx + x(3u^2 + 1)du = 0,$$

or

$$\frac{dx}{x} + \frac{1}{4}\left\{\frac{1}{u} + \frac{2u}{u^2 + 1}\right\} du = 0.$$

Integration of both sides then gives

$$\ln\left\{x^4|u|(u^2 + 1)\right\} = c_1,$$

or

$$\ln\left\{x^4\left|\frac{y}{x}\right|\left(\left(\frac{y}{x}\right)^2 + 1\right)\right\} = c_1,$$

or

$$xy(x^2 + y^2) = c,$$

where $c \in \mathbb{R}$ since by inspection $y = 0$ is a trivial solution of the DE, is the general solution of the DE.

3. Find the general solution of the differential equation

$$\cos x\, dx + \left(1 + \frac{2}{y}\right) \sin x\, dy = 0$$

by finding an appropriate integrating factor. (15 points)

Solution. Let $M = \cos x$ and $N = (1 + 2/y)\sin x$. Since

$$M_y = 0 \neq \left(1 + \frac{2}{y}\right) \cos x = N_x,$$

the DE is not exact. But

$$\frac{M_y - N_x}{N} = -\cot x;$$

hence the DE has integrating factor

$$\mu(x) = \exp^{\int (-\cot x) dx} = \frac{1}{\sin x}.$$

Multiplying this with the DE yields the equation

$$\cot x\, dx + \left(1 + \frac{2}{y}\right) dy = 0$$

which is exact and separable. Integration of both sides gives

$$\ln|\sin x| + y + \ln y^2 = c_1,$$

or

$$y^2 e^y \sin x = c,$$

where $c \neq 0$ since $y = 0$ is not a solution of the DE.

4. Find the general solution of the differential equation

$$e^{-x}y'' - 2e^{-x}y' + 2e^{-x}y = \tan x.$$

(15 points)

Solution. Write the DE as

$$y'' - 2y' + 2y = e^x \tan x.$$

which is a linear nonhomogeneous DE of constant coefficients with auxiliary equation

$$m^2 - 2m + 2 = 0$$

whose roots are $m = 1 \pm i$. Thus the complementary solution is

$$y_c = e^x(c_1 \cos x + c_2 \sin x); \quad x \in \mathbb{R},$$

By the variation of parameters method a particular solution of the DE is

$$y_p = e^x(u_1 \cos x + u_2 \sin x); \quad x \in \mathbb{R},$$

where $u_1' = W_1/W$, $u_2' = W_2/W$,

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix} = e^{2x},$$

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ e^x \tan x & e^x(\cos x + \sin x) \end{vmatrix} = -e^{2x} \sin x \tan x,$$

and

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x(\cos x - \sin x) & e^x \tan x \end{vmatrix} = e^{2x} \sin x.$$

Thus $u_1' = -\sin x \tan x$ and $u_2' = \sin x$; hence $u_1 = \ln|\sec x - \tan x| + \sin x$ and $u_2 = -cosx$. Thus

$$y_p = e^x \cos x \ln|\sec x - \tan x|; \quad x \in \mathbb{R},$$

and the general solution of the DE is

$$y = e^x(c_1 \cos x + c_2 \sin x) + e^x \cos x \ln|\sec x - \tan x|.$$

5. Solve by two different methods the Cauchy-Euler differential equation

$$x^3y''' + xy' - y = 0.$$

(15 points)

Solution. Method 1. By using the substitution $x = e^t$, we find $y' = y^{\cdot}/x$, $y'' = (y^{\cdot\cdot} - y^{\cdot})/x^2$, and $y''' = (y^{\cdot\cdot\cdot} - 3y^{\cdot\cdot} + 2y^{\cdot})$, where $y^{\cdot} = dy/dt$, $y^{\cdot\cdot} = d^2y/dt^2$, and $y^{\cdot\cdot 0} = d^3y/dt^3$. and the differentia equation becomes

$$(y^{\cdot\cdot\cdot} - 3y^{\cdot\cdot} + 3y^{\cdot} - y = 0$$

which is a homogenous linear DE with constant coefficients whose auxiliary equation is $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$ and a single root $m = 1$ repeated thrice. Thus the general solution of the differential equation is

$$y = (c_1 + c_2t + c_3t^2)e^t = (c_1 + c_2 \ln x + c_3(\ln x)^2)x, \quad x > 0.$$

Method 2. By using the substitution $y = x^m$ the differential equation becomes

$$x^3(m^3 - 3m^2 + 3m - 1) = 0,$$

or for $x > 0$,

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0,$$

which is the auxiliary equation for the DE. Thus the DE has a fundamental set $\{x, x\ln x, x(\ln x)^2\}$ and the general solution of the differential equation is

$$y = (c_1 + c_2 \ln x + c_3(\ln x)^2)x, \quad x > 0.$$

6. Find the general solution of the differential equation

$$y''' - 2y'' - 4y' + 8y = 6xe^{2x}$$

by using the **method of undetermined coefficients**.
(Hint: m=2)

(15 points)

Solution. This is a homogenous linear DE with constant coefficients whose auxiliary equation is

$$m^3 - 2m^2 - 4m + 8 = (m-2)^2(m+2) = 0$$

and whose roots are $m = 2(twice)$ and $m = -2$. Thus the complementary solution is

$$y_c = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}.$$

By the method of undetermined coefficients the particular solution has the form

$$y_p = x^2(ax+b)e^{2x}.$$

To find a and b we write

$$y_p' = [2ax^3 + (3a+2b)x^2 + 2bx]e^{2x},$$

$$y_p'' = [4ax^3 + 4(3a+b)x^2 + (6a+8b)x + 2b]e^{2x}],$$

and

$$y_p''' = [8ax^3 + (36a+8b)x^2 + (36a+24b)x + 6a + 12b]e^{2x}].$$

Then by substitution in the differential equation we obtain

$$[24ax + (6a+8b)]e^{2x} = 6xe^{2x}, \quad \text{or} \quad 24ax + (6a+8b) = 6x.$$

Hence, $24a = 6$ and $6a + 8b = 0$. Thus $a = 1/4$, $b = -3/16$, and

$$y_p = x^2(x/4 - 3/16)e^{2x}.$$

Therefore the general solution is

$$y = [c_1 + c_2 x - 3/16x^2 + x^3/4]e^{2x} + c_3 e^{-2x}.$$

7. Consider a linear homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad -\infty < x < \infty,$$

where a_2, a_1, and a_0 are continuous functions with $a_2(x) \neq 0$ for all x.

(a) Can a set $\{y_1, y_2\}$, where $y_1(0) = 1$, $y_2(0) = -1$, $y_1'(0) = -1$, $y_2'(0) = 1$, be a fundamental set of solutions for the differential equation? **Justify your answer**.

(5 points)

Solution. NO since $W(y_1(0), y_2(0)) = 0$.

(b) Can a set $\{y_1, y_2, y_3\}$ of three nontrivial solutions of the differential equation be linearly independent? **Justify your answer**.

(5 points)

Solution. NO since a fundamental set consists only of two solutions; thus one solution must be a linear combination of the other two.