Math 202 — Spring 2006
Differential Equations, sections 1–4
Quiz 1, March 7 — Duration: 1 hour

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation Th 12:30	Recitation Th 2	Recitation F 10	Recitation F 9
Ms. Jaber	Ms. Jaber	Prof. Makdisi	Prof. Makdisi

INSTRUCTIONS:

1. Write your NAME and AUB ID number above, and circle your SECTION.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.
3. You may use the back of each page for scratchwork OR for solutions. There are two extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. It is okay to leave the solution of any differential equation in implicit form.
5. Do as much of the exam as you can, and budget your time carefully. If you cannot do a certain integral, just leave it as an integral in your solution for partial credit on the rest of the problem.
6. No calculators, books, or notes allowed. Turn off and put away any cell phones or beepers.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
The problems are repeated inside the booklet — PLEASE
SOLVE EACH PROBLEM ON ITS CORRESPONDING PAGE INSIDE.

1. (12 pts) Find the general solution of $y' = -2(x+2y)^2$.

2. (12 pts) Find the solution of the initial-value problem

$$y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 2.$$

3. (12 pts) Find the general solution of $y' + (\tan 2x)y = \dfrac{y^3}{1+\sin 2x}$.

4. (12 pts) Given that $y_1 = e^{3x}$ is a particular solution of the following differential equation, find its general solution:

$$xy'' - (3x+1)y' + 3y = 0.$$

5. (12 pts) Find the general solution of:

$$\left(\ln x - \frac{y^2}{2}\right)dx + \left(x^2 + xy\right)dy = 0.$$

6. a) (6 pts) On what interval is a unique solution to the following initial-value problem (IVP) guaranteed to exist?

$$x^2(x-4)y'' + (\tan x)y' + \sqrt{x+1}\, y = 0, \qquad y(3) = 5, \qquad y'(3) = 7.$$

b) (UNRELATED, 6 pts) Find **two distinct** solutions to the IVP

$$xy' = 2y, \qquad y(1) = 1.$$

Both solutions must be continuous and differentiable for all $x \in \mathbf{R}$.

(Finding one solution is easy, and you should all do so for 2 points. Finding a second solution is a challenge worth 4 points.)

1. (12 pts) Find the general solution of $y' = -2(x+2y)^2$.

This is a DE of the form $y' = F(ax+by+c)$.

We substitute $u = x+2y$ and get

$$\frac{du}{dx} = 1 + 2\frac{dy}{dx} = 1 + 2\left(-2u^2\right) = 1 - 4u^2$$

so $\frac{du}{1-4u^2} = dx$, and $\int \frac{du}{1-4u^2} = \int dx + C = x + C$.

To integrate, we do partial fractions:

$$\frac{1}{1-4u^2} = \frac{1}{(1-2u)(1+2u)} = \frac{A}{1-2u} + \frac{B}{1+2u} \quad \text{where } A(1+2u) + B(1-2u) = 1$$

$$\iff \begin{cases} A + B = 1 & (\text{coeff of } 1) \\ 2A - 2B = 0 & (\text{coeff of } u) \end{cases} \iff A = B = \tfrac{1}{2}.$$

$$\text{so } \int \frac{du}{1-4u^2} = \int \left(\frac{\frac{1}{2}}{1-2u} + \frac{\frac{1}{2}}{1+2u}\right) du$$

$$= \int \frac{-\frac{1}{4}\, d(1-2u)}{1-2u} + \int \frac{\frac{1}{4}\, d(1+2u)}{1+2u}$$

$$= -\tfrac{1}{4}\ln|1-2u| + \tfrac{1}{4}\ln|1+2u|.$$

combining with the above & substituting $u = x+2y$, we obtain the general implicit solution

$$\boxed{-\tfrac{1}{4}\ln|1-2(x+2y)| + \tfrac{1}{4}\ln|1+2(x+2y)| = x + C}$$

(N.B. you can also exponentiate to obtain

$$\frac{1+2x+4y}{1-2x-4y} = Ae^{+4x}$$

this can be made explicit by solving for y...)

2. (12 pts) Find the solution of the initial-value problem

$$y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 2.$$

This is a constant-coefficient homogeneous linear differential equation with auxiliary equation $m^3 + m^2 + 4m + 4 = 0$.

Trying factors of 4 ($\pm1, \pm2, \pm4$), we see that $m=-1$ is a root $\left[(-1)^3+(-1)^2+4(-1)+4 = -1+1-4+4=0\right]$

so $m+1$ is a factor, and $m^3+m^2+4m+4 = (m+1)(m^2+4)$ either directly or by division of polynomials.

Thus the roots of the auxiliary equation are
$m_1 = -1$
$m_2 = 2i$
$m_3 = -2i$ $\Big\}$ from $m^2+4=0 \iff m^2 = \pm\sqrt{-4} = \pm 2\sqrt{-1}$

giving rise to real solutions $y_1 = e^{-x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$. The general soln is $\underline{y = Ae^{-x} + B\cos 2x + C\sin 2x}$

Now substitute in the initial values to solve for A, B, C:

$$y = Ae^{-x} + B\cos 2x + C\sin 2x \implies y(0) = A + B = 0$$
$$y' = -Ae^{-x} - 2B\sin 2x + 2C\cos 2x \implies y'(0) = -A + 2C = 1$$
$$y'' = Ae^{-x} - 4B\cos 2x - 4C\sin 2x \implies y''(0) = A - 4B = 2.$$

we solve this system (e.g. by Gaussian elimination, see below) & get $\begin{cases} A = 2/5 \\ B = -2/5 \\ C = 7/10 \end{cases}$

so the solution is $\boxed{y = \frac{2}{5}e^{-x} - \frac{2}{5}\cos 2x + \frac{7}{10}\sin 2x}$

Gaussian elimination, pretending we are robots (in practice, we would use some shortcuts):

$$\begin{cases} A + B = 0 & (1) \\ -A + 2C = 1 & (2) \\ A - 4B = 2 & (3) \end{cases} \iff \begin{cases} A + B = 0 & (1) \\ B + 2C = 1 & (2)+(1)=(2') \\ -5B = 2 & (3)-(1)=(3') \end{cases} \iff \begin{cases} A - 2C = -1 & (1)-(2') = (1'') \\ B + 2C = 1 & (2') \\ 10C = 7 & (3')+5(2') = (3'') \end{cases}$$

$$\iff \begin{cases} A - 2C = -1 & (1'') \\ B + 2C = 1 & (2') \\ C = 7/10 & \tfrac{1}{10}\cdot(3'') = (3''') \end{cases} \iff \begin{cases} A = 2/5 & (1'') + 2\cdot(3''') \\ B = -2/5 & (2') - 2\cdot(3''') \\ C = 7/10 & (3''') \end{cases}$$

3. (12 pts) Find the general solution of $y' + (\tan 2x)y = \dfrac{y^3}{1+\sin 2x}$.

This equation has the form $y' + P(x)y = Q(x)y^3$, so it is <u>Bernoulli</u> with $n=3$ & we solve it by substituting $u = y^{1-3} = y^{-2}$. This implies $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$.

Multiply the original DE by $-2y^{-3}$ to obtain:

$$-2y^{-3}y' - 2y^{-3+1}(\tan 2x) = \frac{-2y^{-3+3}}{1+\sin 2x}$$

$$\Leftrightarrow \quad \frac{du}{dx} - 2u(\tan 2x) = \frac{-2}{1+\sin 2x} \quad (*)$$

linear DE in u, with integrating factor $\mu = e^{\int -2\tan 2x\,dx}$

calculate $\mu = e^{\int \frac{-2\sin 2x}{\cos 2x}dx} = e^{\int \frac{d(\cos 2x)}{\cos 2x}} = e^{\ln \cos(2x)} = \cos 2x$

multiply $(*)$ by $\cos 2x$ to obtain

$$u'(\cos 2x) - 2u(\sin 2x) = \frac{-2\cos 2x}{1+\sin 2x}$$

$$\Leftrightarrow \quad [u\cos 2x]' = \frac{-2\cos 2x}{1+\sin 2x}$$

$$\Leftrightarrow \quad u\cos 2x = \int \frac{-2\cos 2x\,dx}{1+\sin 2x} + C$$

$$= \int -\frac{d(1+\sin 2x)}{1+\sin 2x} + C$$

$$= -\ln(1+\sin 2x) + C$$

[note $|1+\sin 2x| = 1+\sin 2x$]

so $\boxed{y^{-2} = u = \dfrac{-\ln(1+\sin 2x) + C}{\cos 2x}}$ implicit solution

(or $y = \pm\left(\dfrac{-\ln(1+\sin 2x) + C}{\cos 2x}\right)^{-1/2}$)

explicit soln, if you like

4. (12 pts) Given that $y_1 = e^{3x}$ is a particular solution of the following differential equation, find its general solution:

$$xy'' - (3x+1)y' + 3y = 0.$$

This is a problem on reduction of order. We put $y = uy_1 = ue^{3x}$

so
$$\begin{cases} y = ue^{3x} & \longleftarrow \text{mult. by } 3 \\ y' = 3ue^{3x} + u'e^{3x} & \longleftarrow \text{mult. by } -(3x+1) \\ y'' = 9ue^{3x} + 6u'e^{3x} + u''e^{3x} & \longleftarrow \text{mult. by } x \end{cases}$$

add

$$0 = xy'' - (3x+1)y' + 3y = u\begin{bmatrix} 3e^{3x} \\ -(3x+1)\cdot 3e^{3x} \\ +9xe^{3x} \end{bmatrix} + u'\begin{bmatrix} 6xe^{3x} \\ - \\ (3x+1)e^{3x} \end{bmatrix} + u''\left[xe^{3x}\right]$$

$$= \underset{(\text{check})}{u\cdot[0]} + u'\left[(3x-1)e^{3x}\right] + u''\left[xe^{3x}\right]$$; can cancel e^{3x} (not identically zero, in fact it's never zero!)

$\Longleftrightarrow$ $0 = u'\cdot(3x-1) + u''\cdot(x)$. Now substitute $w = u'$ so $w' = u''$

& get $xw' + (3x-1)w = 0$ (can solve by integrating factor or by separating)

$\Longleftrightarrow$ $\frac{dw}{w} = \frac{(1-3x)dx}{x}$ $\Rightarrow$ $\int \frac{dw}{w} = \int \frac{(1-3x)dx}{x} \underline{\underline{+C}}$

so $\ln|w| = \int \frac{dx}{x} - \int 3dx + C = \ln|x| - 3x + C$

& $w = Axe^{-3x}$

[Many of you made a mistake in exponentiating: always remember $e^{\ln|x| - 3x + C} = |x| \cdot e^{-3x} \cdot e^{C}$ $\underline{\text{times}}$, $\underline{\underline{\text{NOT}}}$ $\underline{\underline{\text{PLUS}}}$!]

so $u' = Axe^{-3x}$, and

$u = \int u'dx \underline{\underline{+B}} = A\int xe^{-3x}dx + B$ (now integrate by parts)

$$= A\cdot\int x\, d\frac{(e^{-3x})}{-3} + B$$

$$= A\cdot\left[\frac{xe^{-3x}}{-3} - \int \frac{e^{-3x}}{-3}dx\right] + B$$

$$= A\left[-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right] + B.$$

$$\boxed{y = ue^{3x} = A\cdot\left[\frac{-x}{3} - \frac{1}{9}\right] + Be^{3x}}$$
$(= A'(3x+1) + Be^{3x})$

note that the final solution is $y = Ay_2 + By_1$, $y_1 = e^{3x}$, y_2 "new" soln

5. (12 pts) Find the general solution of:

$$\left(\ln x - \frac{y^2}{2}\right)dx + \left(x^2 + xy\right)dy = 0.$$

None of our other tricks works, and the equation is not exact to begin with (check this!), so we try multiplying by an integrating factor $\mu = \mu(x,y)$:

$$\underbrace{\mu(x,y)\cdot\left[\ln x - \frac{y^2}{2}\right]}_{M} dx + \underbrace{\mu(x,y)\cdot\left[x^2 + xy\right]}_{N} dy = 0$$

we want $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, i.e.

$$\frac{\partial \mu}{\partial x}\cdot(x^2+xy) + \mu\cdot(2x+y) = \frac{\partial \mu}{\partial y}\cdot\left(\ln x - \frac{y^2}{2}\right) + \mu\cdot(-y)$$

[move to LHS]

$$\iff \frac{\partial \mu}{\partial x}\cdot(x^2+xy) + \mu\cdot(2x+2y) = \frac{\partial \mu}{\partial y}\cdot\left(\ln x - \frac{y^2}{2}\right) \quad (*)$$

try $\mu = \mu(x)$ so $\frac{\partial \mu}{\partial y} = 0$ & $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$

[NOTE many of you wrote $\mu(x)$ to mean both μ a function of x alone & $\frac{\partial \mu}{\partial x}$, which should be written with a subscript: μ_x]

so $(*)$ becomes: $\frac{d\mu}{dx}\cdot(x^2+xy) + \mu\cdot(2x+2y) = 0$

so $\frac{d\mu}{dx}\cdot x\cdot\cancel{(x+y)} + \mu\cdot 2\cdot\cancel{(x+y)} = 0$

$$\iff x\frac{d\mu}{dx} + 2\mu = 0 \iff \frac{d\mu}{\mu} = -2\frac{dx}{x}$$

[note it is very important to have all the y's cancel; this equation must be only in terms of μ & x since $\mu = \mu(x)$.]

Now integrate to get only one μ: $\ln\mu = -2\ln x$
(that is all we need)

so $\boxed{\mu = \frac{1}{x^2}}$.

multiply the original equation by μ; it becomes

$$\boxed{\left[\frac{\ln x}{x^2} - \frac{y^2}{2x^2}\right]dx + \left[1 + \frac{y}{x}\right]dy = 0 \quad (**)}$$

this is now exact!

[CONTINUED ON P. 7]

6. a) (6 pts) On what interval is a unique solution to the following initial-value problem (IVP) guaranteed to exist?

$$x^2(x-4)y'' + (\tan x)y' + \sqrt{x+1}\, y = 0, \qquad y(3) = 5, \qquad y'(3) = 7.$$

This is a linear DE, $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, so there is guaranteed to exist a unique solution on the largest possible interval containing $x_0 = 3$ and not containing any bad points. Here bad points are those where any of a_2, a_1, or a_0 is discontinuous, or where $a_2(x) = 0$.

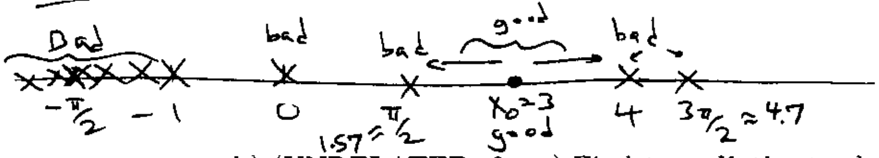
$a_2(x) = 0$: $x = 0$ or $x = 4$

$a_2(x)$ discontinuous: no bad pts from this

$a_1(x)$ discontinuous: $\tan x$ is discontinuous at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

(i.e. $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$)

$a_0(x)$ discontinuous: at $x < -1$ (it's not even defined there)



the interval is $(\pi/2, 4)$ i.e. $\frac{\pi}{2} < x < 4$.

b) (UNRELATED, 6 pts) Find **two distinct** solutions to the IVP

$$xy' = 2y, \qquad y(1) = 1.$$

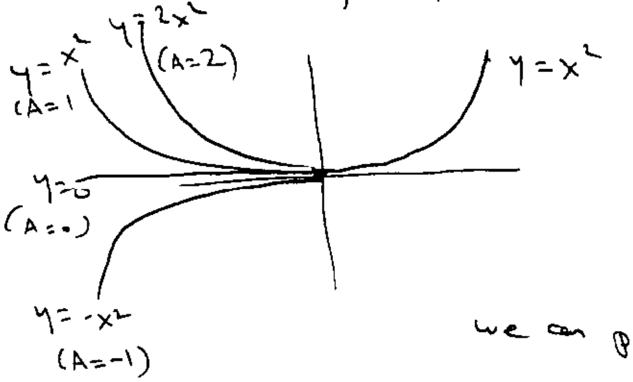
Both solutions must be continuous and differentiable for all $x \in \mathbf{R}$.

(Finding one solution is easy, and you should all do so for 2 points. Finding a second solution is a challenge worth 4 points.)

1st solution: separate $\frac{dy}{y} = 2\frac{dx}{x}$ so, integrating, $\ln|y| = 2\ln|x| + C$ & $y = Ax^2$ "general soln".

the initial value $y(1) = 1$ gives $\boxed{y_1 = x^2}$ 1st solution.

2nd solution: this eqn is linear & has a bad pt at $x = 0$ (check) so the IVP has a unique soln in the good interval $(0, +\infty)$. So any other solution y_2 must satisfy $y_2 = x^2$ for $x > 0$. for $x < 0$ we expect $y_2 = Ax^2$



note that as $x \to 0^+$, $y_2 = x^2 \to 0$

$y_2' = (2x) \to 0$

as $x = 0^-$, $y_2 = Ax^2 \to 0$

$y_2' = 2A_2x \to 0$

we can pick any A we like! The resulting solution is continuous & differentiable for all $x \in \mathbb{R}$ including $x = 0$

For example, let us take

$y_2 = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ OR $y_2 = \begin{cases} x^2, & x \geq 0 \end{cases}$

continuation of exercise 5:

$$\underbrace{\left[\frac{\ln x}{x^2} - \frac{y^2}{2x^2}\right]}_{\substack{\text{this is } f_x \\ \frac{\partial f}{\partial x}}} dx + \underbrace{\left[1 + \frac{y}{x}\right]}_{\substack{\text{this is} \\ \frac{\partial f}{\partial y} = f_y}} dy = 0$$

(xx) copied exact eqn

for some f

It's easiest to integrate f_y first:

$$f = \int_{\substack{\text{pretending} \\ x \text{ is} \\ \text{constant}}} \left[1 + \frac{y}{x}\right] dy = y + \frac{y^2}{2x} + g(x)$$

substitute into $f_x = \frac{\ln x}{x^2} - \frac{y^2}{2x^2}$ + get

$$0 - \cancel{\frac{y^2}{2x^2}} + \frac{dg}{dx} = \frac{\ln x}{x^2} - \cancel{\frac{y^2}{2x^2}}, \text{ so } \frac{dg}{dx} = \frac{\ln x}{x^2}$$

$$g = \int \frac{\ln x}{x^2}\, dx = \int \ln x \; d\left(-\frac{1}{x}\right)$$

now integrate by parts (we only need one g, so don't include + constant)

$$= -\frac{1}{x}\ln x - \int -\frac{1}{x} \cdot \frac{dx}{x} = -\frac{1}{x}\ln x + \int \frac{dx}{x^2} = -\frac{1}{x}\ln x - \frac{1}{x}$$

so $$\boxed{f = y + \frac{y^2}{2x} - \frac{1}{x}\ln x - \frac{1}{x}}$$

The solution is $f(x,y) = C$, i.e.

$$\boxed{y + \frac{y^2}{2x} - \frac{1}{x}\ln x - \frac{1}{x} = C}$$