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Math 202 — Spring 2008
Differential Equations, sections 17–20
Quiz 1, March 7 — Duration: 1 hour 10 minutes

JUST A MINUTE — PLEASE READ THE INSTRUCTIONS BELOW FIRST

1. Write your name, AUB ID number, and section number ON THE FRONT COVER OF YOUR AUB EXAMINATION BOOKLET.
To remind you, the sections are as follows:

Section 17	Section 18	Section 19	Section 20
Recitation W 8	Recitation Tu 2	Recitation Tu 3:30	Recitation Tu 5
Prof. Makdisi	Mrs. Karam	Mrs. Karam	Mrs. Karam

2. You may work on the problems in ANY ORDER in your exam booklet, but please make it clear which problem you are solving on any given page. In particular, PLEASE INDICATE IF THE SOLUTION TO A PROBLEM IS CONTINUED ON A LATER PAGE.

3. Do as much of the exam as you can, and budget your time carefully. The problems are arranged APPROXIMATELY in increasing order of difficulty. Take a minute at the start of the exam to decide which problems to work on first and to plan strategy.

4. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points, except for problems 1 and 6, and there are 6 problems for a TOTAL 72 points.

5. If you cannot do a certain integral, just leave it as an integral in your solution for partial credit on the rest of the problem. Also do NOT bother to simplify fractions; if the answer to a problem is $3e^2(17/6 + 5/11 - 7/4)$, just leave it that way.

6. No calculators, books, or notes allowed. Turn off and put away any cell phones or beepers.

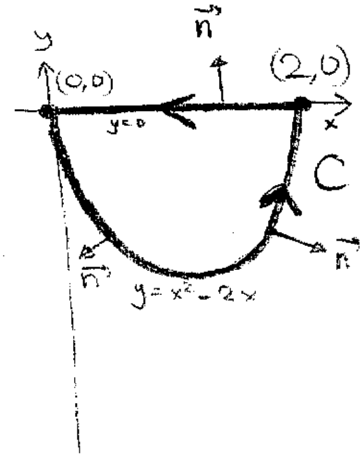
GOOD LUCK!

(Remember, each problem is worth 12 points, except for problems 1 and 6, for a total of 72)

1. (8 pts) Find the solution to the initial value problem

$$dy/dx = 2xy\sin(\pi - x^2), \qquad y(0) = 10.$$

2. Let C be the closed curve in the plane shown at right; part of C is the parabola $y = x^2 - 2x$ from $(0,0)$ to $(2,0)$, and part is the line segment from $(2,0)$ to $(0,0)$. Let $\vec{\mathbf{F}}$ be the vector field



$$\vec{\mathbf{F}} = (xy, 0).$$

a) (6 pts) Compute the work integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ by directly parametrizing (both parts of) C. Remember instruction 5 above.

b) (3 pts) Set up **but do not evaluate** the double integral that is equal to $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ by Green's theorem.

c) (3 pts) Set up **but do not evaluate** the double integral that is equal to $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$ by Green's theorem.

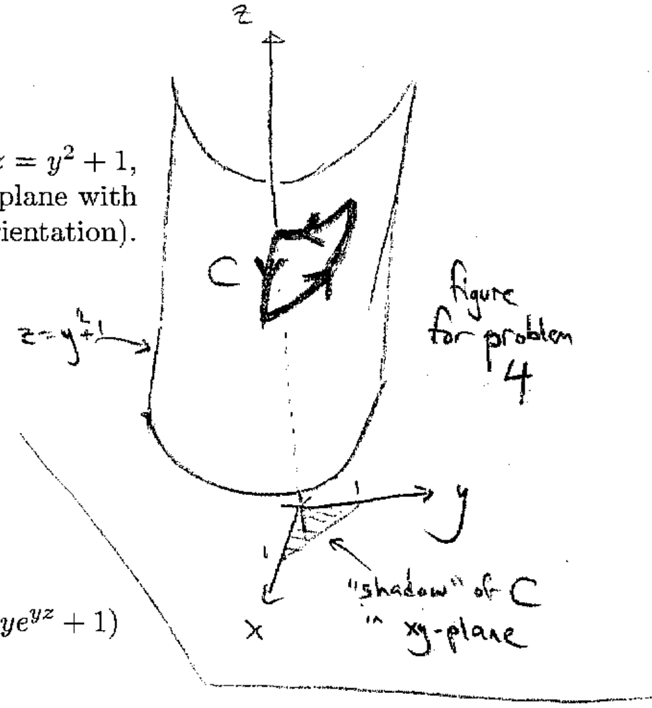
3. Consider the parametrized surface S given by $\vec{\mathbf{r}}(u,w) = (u, u\cos w, u\sin w)$, with $0 \le u \le 1$ and $0 \le w \le 2\pi$.

a) (4 pts) Make a rough sketch of S (your picture should also show the x, y, and z coordinate axes).

b) (8 pts) Find the surface area of S.

4. (12 pts) Let C be the curve lying on the surface $z = y^2 + 1$, whose sides lie vertically above the triangle in the xy-plane with $x = 0$, $y = 0$, $x + y = 1$. (See diagram for C and its orientation). Use **Stokes' theorem** to compute

$$\int_C 3xz\,dx + y\,dy + x^2\,dz.$$



Remember instruction 5!

5. a) (5 pts) Given that the vector field

$$\vec{\mathbf{F}} = (2xy, x^2 + ze^{yz}, ye^{yz} + 1)$$

is conservative, find a potential function for $\vec{\mathbf{F}}$.

b) (3 pts) Use part (a) to compute $\displaystyle\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$ where C is the parametrized curve given by

$$\vec{\mathbf{r}}(t) = (\sin(\pi t^3), t^4, t^5 + t^6), \qquad 0 \le t \le 1.$$

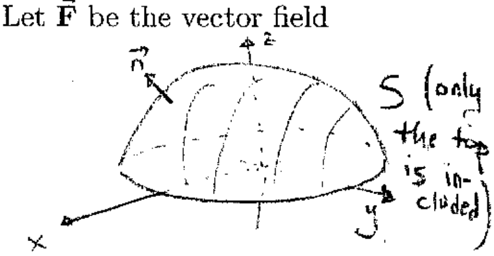
c) (4 pts) Show that the vector field

$$\vec{\mathbf{G}} = (2xy + z, x^2 + ze^{yz}, ye^{yz} + 1)$$

is **not** conservative.

6. Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, $\quad z \ge 0$. **The flat "bottom" is NOT included.** Orient S with a normal vector that points away from the origin. Let $\vec{\mathbf{F}}$ be the vector field

$$\vec{\mathbf{F}} = (2xz, 0, 1).$$



a) (6 pts) Set up but **do not compute** the flux integral

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,d\sigma$$

in terms of an integral in the xy-plane. Remember to specify the region of integration.

b) (10 pts) Use the divergence theorem in an appropriate way to compute $\displaystyle\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,d\sigma$.

Math 202 – Spring 2008 (1)

(M)

Sections 17-20, Professor Khuri-Makdisi

Solutions to Quiz 1

1. Separate the variables:

$$\frac{dy}{y} = 2x\sin(\pi - x^2)\,dx$$

now integrate both sides:

$$\ln|y| = \int \frac{dy}{y} = \int 2x\sin(\pi-x^2)\,dx + C = -\int \sin(\pi-x^2)\,d(\pi-x^2) \quad + C$$

$$= +\cos(\pi-x^2) + C$$

so $y = \pm e^{C}\cdot e^{\cos(\pi-x^2)} = A e^{\cos(\pi-x^2)}$. [new constant $A=\pm e^C$, may be positive, negative, or even 0]

Use the initial condition $y(0)=10$ to conclude $Ae^{\cos(\pi-0)}=10$

$\Rightarrow A = \frac{10}{e^{\cos\pi}} = \frac{10}{e^{-1}} = 10e$, so the solution is

$$\boxed{y = 10e\cdot e^{\cos(\pi-x^2)}}$$

2. a) $\oint_C \vec F\cdot d\vec r = \int_{C_1}\vec F\cdot d\vec r + \int_{C_2}\vec F\cdot d\vec r.$

We parametrize C_1 by $\vec r(t) = (t, t^2-2t)$ for $0\le t\le 2$; so $\frac{d\vec r}{dt} = (1, 2t-2)$

and C_2 by $\vec r(t) = (2-t, 0)$ for $0\le t\le 2$ (note the orientation!)

so $\frac{d\vec r}{dt} = (-1,0)$ for C_2.

Then our answer is

$$\int_{t=0}^{2} (xy,0)\Big|_{(x,y)=(t,t^2-2t)} \cdot (1,2t-2)\,dt + \int_{t=0}^{2} (xy,0)\Big|_{(x,y)=(2-t,0)} \cdot(-1,0)\,dt$$

$$= \int_{t=0}^{2} (t^3-2t^2, 0)\cdot(1,2t-2)\,dt + \int_{t=0}^{2}(0,0)\cdot(-1,0)\,dt = \int_{t=0}^{2}(t^3-2t^2)\,dt + 0 = \dots = \boxed{\frac{-4}{3}}$$

(ok to leave it as $\frac{16}{4}-\frac{16}{3}$)

(2)

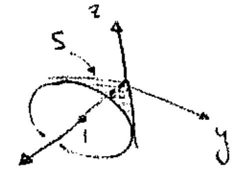
b) $M = xy$, $N=0$, C is oriented counterclockwise

$$\Rightarrow \oint_C \vec F\cdot d\vec r = \iint_R (N_x - M_y)\,dA = \boxed{\int_{x=0}^{2}\int_{y=x^2-2x}^{0} (-x)\,dy\,dx}$$

c) $\vec n$ points outwards so

$$\oint_C \vec F\cdot\vec n\,ds = \iint_R (M_x+N_y)\,dA = \boxed{\int_{x=0}^{2}\int_{y=x^2-2x}^{0} y\,dy\,dx}$$

3. a) we have $\underbrace{x^2 = y^2+z^2}_{\text{cone with rotational symmetry about the x-axis}}$ and $0\le x\le 1$, so S is



(the side of the cone only)

side view of S:

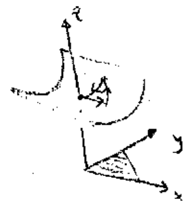
b) $$d\sigma = \left|\frac{\partial\vec r}{\partial u}\times\frac{\partial\vec r}{\partial w}\right| du\,dw = \left|(1,\cos w,\sin w)\times(0,-u\sin w, u\cos w)\right| du\,dw$$

$$= \left|(u, -u\cos w, -u\sin w)\right| du\,dw$$ [use $\det\begin{pmatrix}\vec i & \vec j & \vec k\\ 1 & \cos w & \sin w\\ 0 & -u\sin w & u\cos w\end{pmatrix}$]

$$= u\sqrt2\,du\,dw$$ [technically we should write $|u|\sqrt2\,du\,dw$, but $u\ge 0$]

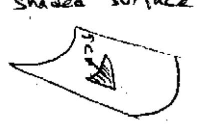
So the surface area is $\int_{u=0}^{1}\int_{w=0}^{2\pi} u\sqrt2\,du\,dw = \dots = \boxed{\pi\sqrt2}$.

4. side view of C:



since C is oriented counterclockwise viewed from above, it corresponds to the orientation of the surface $z = y^2+1$ with an upwards pointing normal vector.

specifically, C is the oriented boundary of the shaded surface S below which is a "triangular" part of $z=y^2+1$:



(†)

Stokes' theorem says $\displaystyle\int_C \vec F\cdot d\vec r = \iint_S (\vec\nabla\times\vec F)\cdot\vec n\,d\sigma$

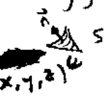
Here $\vec F=(3xz,y,x^2)$ so $\vec\nabla\times\vec F = \det\begin{pmatrix}\vec i & \vec j & \vec k\\ \partial_x & \partial_y & \partial_z\\ 3xz & y & x^2\end{pmatrix} = (0,-2x+3x,0) = (0,x,0)$

S is parametrized by $z=y^2+1=f(x,y)$, so $\vec n\,d\sigma = \underset{\substack{\vec n \text{ has}\\ \text{positive } z-\\ \text{component}}}{+}(-f_x,-f_y,1)\,dx\,dy$
$(\vec r(x,y)=(x,y,f(x,y)))$ $\qquad = (0,-2y,1)\,dx\,dy$

So finally our desired answer is

$$\iint_S (0,x,0)\cdot\vec n\,d\sigma = \iint_{(x,y)\in\text{ projection of } S} (0,x,0)\Big|_{z=y^2+1}\cdot(0,-2y,1)\,dx\,dy$$

(x,y,z)



$$= \int_{y=0}^{1}\int_{x=0}^{1-y} -2xy\,dx\,dy = \dots = \underbrace{-\tfrac12+\tfrac23-\tfrac14}_{\text{you can leave the answer like this.}} = \boxed{-\tfrac{1}{12}}$$

5 a) we are given that there exists f with $\vec\nabla f=\vec F$. Hence

$f_x = 2xy$ (i) $\Longrightarrow$ $f = x^2y + k(y,z)$ "integrate (i) with respect to x, holding y & z constant"

$f_y = x^2+ze^{yz}$ (ii)

$f_z = ye^{yz}+1$ (iii)

Substitute into (ii): $x^2+\frac{\partial k}{\partial y} = x^2+ze^{yz}$

$\Longrightarrow \frac{\partial k}{\partial y} = ze^{yz}$ Note there is no x left. Good.

$\Longrightarrow k = \int_{z\text{ const.}} ze^{yz}\,dy = e^{yz}+l(z)$

$\Longrightarrow f = x^2y+e^{yz}+l(z)$

now substitute into (iii), to get

$0+ye^{yz}+\frac{dl}{dz} = ye^{yz}+1 \Rightarrow \frac{dl}{dz}=1$. We can choose $l(z)=z$ (WE ONLY NEED ONE f)

to obtain $\boxed{f=x^2+e^{yz}+z \text{ is a potential function for } \vec F}$

b) $\displaystyle\int_C \vec\nabla f\cdot\vec T\,ds = f(\text{end point}) - f(\text{start point})$.

Here $f(x,y,z) = x^2+e^{yz}+z$.

The start point is $\vec r(0) = (\sin 0,0,0) = (0,0,0)$
The end point is $\vec r(1) = (\sin\pi,1,2) = (0,1,2)$

so the answer is $f(0,1,2) - f(0,0,0)$
$= 0+e^2+2-0-e^0-0 = e^2+2-1 = \boxed{e^2+1}$.

c) write $\vec G=(M,N,P)$ with $M=2xy+z$
$N = x^2+ze^{yz}$
$P = ye^{yz}+1$.

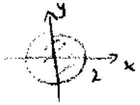
Note that $M_z=1$ while $P_x=0$; thus $M_z\neq P_x$ so the necessary condition(s) for $\vec G$ to be conservative do not hold. (If $\vec G=\vec\nabla g$, this would say $g_{xz}\neq g_{zx}$.)

[Question how can one easily see, by comparing $\vec G$ with $\vec F$, that one must consider M_z to show that $\vec G$ is not conservative?]

6. a) S can be parametrized, using $z=\underset{=f(x,y)}{+\sqrt{4-x^2-y^2}}$ (note $z\geq 0$)

so $\vec n\,d\sigma = \underset{\substack{\text{the } z\text{-component of}\\ \vec n \text{ is positive}}}{+}(-f_x,-f_y,1)\,dx\,dy = \left(\frac{+x}{\sqrt{4-x^2-y^2}},\frac{+y}{\sqrt{4-x^2-y^2}},1\right)dx\,dy$.

The (x,y) region that is the projection of S is the disk:



in the xy-plane.

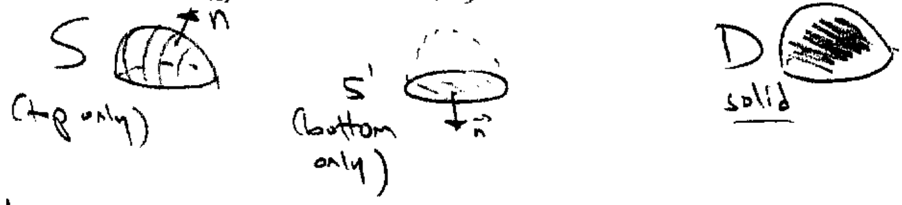
Our answer is hence $\displaystyle\iint_{(x,y)\in\text{disk}} (2xz,0,1)\Big|_{z=\sqrt{4-x^2-y^2}}\cdot\left(\frac{x}{\sqrt{4-x^2-y^2}},\frac{y}{\sqrt{4-x^2-y^2}},1\right)dx\,dy$

$$=\iint_{(x,y)\in\text{disk}}\left(2x\sqrt{4-x^2-y^2}\cdot\frac{x}{\sqrt{4-x^2-y^2}}+0+1\right)dx\,dy$$

$$=\boxed{\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x^2+1)\,dy\,dx} \quad \text{OR} \quad \boxed{\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(2r^2\cos^2\theta+1)\,r\,dr\,d\theta}$$

b) S is not the boundary of a solid region D (indeed, S is not closed). But if we consider S' = the "bottom" flat disk underneath the hemisphere, we can use the divergence theorem to write

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S'} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D (\vec{\nabla}\cdot\vec{F})\,dV.$$



Note the orientation on S' (and on S), pointing <u>outwards</u> from D.

We first calculate $\iint_{S'} \vec{F}\cdot\vec{n}\,d\sigma$, $\vec{n}=(0,0,-1)$: note that on S', $d\sigma = dx\,dy$ because S' is parallel to the xy plane while $\vec{n} = (0,0,-1)$

so $\vec{F}\cdot\vec{n}\Big|_{\text{on } S} = (2xz, 0, 1)\Big|_{z=0} \cdot (0,0,-1) = -1$ and we obtain

$z=0$, not that it matters in this specific example

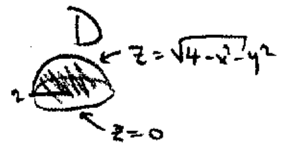
$$\iint_{S'} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S'} -1\,d\sigma = -(\text{area of disk of radius } 2) = \underline{\underline{-4\pi}}.$$

On the other hand, $\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 2z$

and

$$\iiint_D 2z\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{\sqrt{4-r^2}} 2z \cdot r\,dz\,dr\,d\theta$$

(cylindrical is easiest here, but you can also use spherical coordinates or even $dz\,dx\,dy$, using intuition or polar coordinates for the $dx\,dy$ integral over a disk)



$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \left[rz^2\right]_{z=0}^{\sqrt{4-r^2}} dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (r(4-r^2)-0)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (4r - r^3)\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_{r=0}^{2} d\theta = \int_{\theta=0}^{2\pi} [8-4]\,d\theta = \underline{\underline{8\pi}}$$

Finally, $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D - \iint_{S'} = 8\pi - (-4\pi) = \boxed{12\pi}$