Math 202 — Spring 2008
Differential Equations, sections 17–20
Quiz 2, April 11 — Duration: 1 hour

Reminder of instructions. READ AT LEAST NUMBER 5.

1. Write your name, AUB ID number, and section number ON THE FRONT COVER OF YOUR AUB EXAMINATION BOOKLET.
To remind you, the sections are as follows:

Section 17	Section 18	Section 19	Section 20
Recitation W 8	Recitation Tu 2	Recitation Tu 3:30	Recitation Tu 5
Prof. Makdisi	Mrs. Karam	Mrs. Karam	Mrs. Karam

2. You may work on the problems in ANY ORDER in your exam booklet, but please make it clear which problem you are solving on any given page. In particular, PLEASE INDICATE IF THE SOLUTION TO A PROBLEM IS CONTINUED ON A LATER PAGE.

3. Do as much of the exam as you can, and budget your time carefully. The problems are arranged APPROXIMATELY in increasing order of time needed. Take a minute at the start of the exam to decide which problems to work on first and to plan strategy.

4. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points, and there are 6 problems for a TOTAL 72 points.

5. If you cannot do a certain integral, just leave it as an integral in your solution for partial credit on the rest of the problem. YOU MAY LEAVE THE SOLUTION TO A DIFFERENTIAL EQUATION IN IMPLICIT FORM.

6. No calculators, books, or notes allowed. Turn off and put away any cell phones.

GOOD LUCK!

(Remember, each problem is worth 12 points, for a total of 72)

1. Find the general solution of $4xy' + 2y = \dfrac{\cos x}{y^3}$.

2. Find the general solution of $y' = \dfrac{y^4 + x^2y^2 - 9x^4}{xy^3}$.

3. Find the general solution of $(3x^2 + y)\,dx + (x^3 + xy + x)\,dy = 0$.

4. Find the general solution of $y'' - 6y' + 25y = \cos 5x + xe^x$.

5. Solve the initial value problem $y''' - 3y'' + 4y = 0$, $\quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$.

6. We are given that $y_1 = x^{-2}$ is a solution of the homogeneous linear differential equation

$$y'' + 5x^{-1}y' + 4x^{-2}y = 0.$$

a) Use the method of reduction of order to find a second solution y_2 of the above differential equation. **Please assume that** $x > 0$.

b) Show that y_1 and y_2 are linearly independent.

c) Use **variation of parameters** (NOT reduction of order!) to find ONE particular solution (y_P) of the inhomogeneous equation

$$y'' + 5x^{-1}y' + 4x^{-2}y = x^{-1}.$$

$\boxed{1}$ $4xy' + 2y = \dfrac{\cos x}{y^3}$

Bernoulli with $n = -3$

make subst $u = y^{1-n} = y^4$

so $u' = 4y^3 y'$

multiply the DE by $\frac{y^3}{x}$ so that u' appears

$$4y^3 y' + \frac{2y^4}{x} = \frac{\cos x}{x}, \quad \text{so} \quad \boxed{u' + \frac{2}{x}u = \frac{\cos x}{x}}$$

this is a linear DE for u, with integrating factor $\mu = e^{\int \frac{2}{x} dx}$

$= e^{2\ln x} = x^2$.

multiply by μ: $x^2 u' + 2xu = x\cos x$

identify LHS as a derivative: $(x^2 u)' = x\cos x$

integrate: $x^2 u = \int x\cos x \, dx + C = \int x \, d(\sin x) + C$

$$\underbrace{=}_{\int \text{by parts}} x\sin x - \int \sin x \, dx + C = x\sin x + \cos x + C$$

$$\therefore \quad u = \frac{x\sin x + \cos x + C}{x^2} = \frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{C}{x^2}$$

but $u = y^4$

so $\boxed{y^4 = \dfrac{\sin x}{x} + \dfrac{\cos x}{x^2} + \dfrac{C}{x^2}}$ implicit solution.

you can also make this explicit if you like:

$$y = \left(\frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{C}{x^2}\right)^{1/4}$$

$\boxed{2}$ $$y' = \frac{y^4 + x^2y^2 - 9x^4}{xy^3}$$ ← numerator & denominator are both purely of degree 4, so it's a <u>homogeneous</u> DE (in the nonlinear sense: $y' = F(\frac{y}{x})$)

rewrite $$y' = \frac{y^4+x^2y^2-9x^4}{xy^3} \cdot \frac{1/x^4}{1/x^4} = \frac{\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^2 - 9}{\left(\frac{y}{x}\right)^3}$$

substitute $u = \frac{y}{x}$ so $y = xu$ so $y' = xu' + u$. We get

$$xu' + u = \frac{u^4 + u^2 - 9}{u^3}$$

$$\Leftrightarrow x\frac{du}{dx} = \frac{u^4+u^2-9}{u^3} - u = \frac{u^2-9}{u^3}$$

$$\Leftrightarrow \int \frac{u^3\,du}{u^2-9} = \int \frac{dx}{x} + C$$

Note $\deg(u^3) = 3 > \deg(u^2-9) = 2$, so we divide:

$$\begin{array}{r|l} u^3 & u^2-9 \\ -(u^3-9u) & u \\ \hline +9u & \end{array}$$

$$\frac{u^3}{u^2-9} = u + \frac{9u}{u^2-9}$$

$$\Leftrightarrow \int \left(u + \frac{9u}{u^2-9}\right) du = \int \frac{dx}{x} + C$$

you can do this by partial fractions if you like, but it's easier to use $d(u^2-9)$ as a substitution

$$\Leftrightarrow \frac{1}{2}u^2 + \int \frac{9}{2}\frac{d(u^2-9)}{u^2-9} = \ln|x| + C$$

$$\Leftrightarrow \frac{1}{2}u^2 + \frac{9}{2}\ln|u^2-9| = \ln|x| + C$$, but $u = \frac{y}{x}$

$$\Leftrightarrow \boxed{\frac{y^2}{2x^2} + \frac{9}{2}\ln\left|\frac{y^2}{x^2} - 9\right| = \ln|x| + C}$$ general solution.

$\boxed{3}$ as written, the eqn has the form $M\,dx + N\,dy = 0$

with $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2+y) = 1 \neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^3+xy+x) = 3x^2+y+1$.

So,
Multiply by an integrating factor μ & see if we can find μ that makes this exact.

$$\underbrace{\mu\cdot(3x^2+y)}_{\text{new } M}\,dx + \underbrace{\mu\cdot(x^3+xy+x)}_{\text{new } N}\,dy = 0$$

we want $\frac{\partial(\text{new}M)}{\partial y} = \frac{\partial(\text{new}N)}{\partial x}$

so $\mu_y\cdot(3x^2+y) + \mu\cdot(1) = \mu_x\cdot(x^3+xy+x) + \mu\cdot(3x^2+y+1)$

$\iff \mu_y\cdot(3x^2+y) = \mu_x\cdot(x^3+xy+x) + \mu\cdot(3x^2+y)$

It turns out that trying $\mu=\mu(x)$ does not work, but $\mu=\mu(y)$ does work

try $\mu=\mu(y) \underset{so}{\Longrightarrow} \frac{d\mu}{dy}\cancel{(3x^2+y)} = 0 + \mu\cdot\cancel{(3x^2+y)}$

cancel $3x^2+y$

so $\frac{d\mu}{dy} = \mu$; this easily leads to choosing $\boxed{\mu = e^y}$.

So the exact eqn is $\underbrace{e^y(3x^2+y)}_{\text{this is } f_x}\,dx + \underbrace{e^y(x^3+xy+x)}_{\text{this is } f_y}\,dy = 0$

from $\frac{\partial f}{\partial x} = e^y(3x^2+y)$ we get $f = \int e^y(3x^2+y)\,dx + k(y)$
hold y constant

so $\boxed{f = e^y(x^3+xy) + k(y)}$

take $\frac{\partial f}{\partial y} = e^y(x^3+xy) + e^y(x) + \frac{dk}{dy} \overset{\text{want}}{=} e^y(x^3+xy+x)$

we get $\frac{dk}{dy} = 0$, so we can choose $k=0$, so $\boxed{f = e^y(x^3+xy)}$

the general soln is $f(x,y) = C$, so $\boxed{\boxed{e^y(x^3+xy) = C}}$

$\boxed{4}$ $\cdot$ $y'' - 6y' + 25y = \cos 5x + xe^x$

y_c: aux eqn $m^2 - 6m + 25$

roots $m = 3 \pm 4i$

soln $\boxed{y_c = Ae^{3x}\cos 4x + Be^{3x}\sin 4x}$

$y_p = y_{p_1} + y_{p_2}$ where $L[y_{p_1}] = \cos 5x$ & $L(y_{p_2}) = xe^x$

$\underline{y_{p_1}}$ $L[\cos 5x] = 30 \sin 5x$

$L[\sin 5x] = -30 \cos 5x$ $\boxed{y_{p_1} = -\frac{1}{30}\sin 5x}$

$\underline{y_{p_2}}$ $L[xe^x] = (2e^x + xe^x) - 6(e^x + xe^x) + 25xe^x = -4e^x + 20xe^x$

$L[e^x] = e^x - 6e^x + 25e^x = 20e^x$

$$L[Cxe^x + De^x] = (-4C + 20D)e^x + 20Cxe^x \overset{\text{want}}{=} xe^x$$

so $\begin{cases} -4C + 20D = 0 \\ 20C = 1 \end{cases}$ $\rightsquigarrow$ $C = \frac{1}{20}$, $D = \frac{1}{100}$

$\boxed{y_{p_2} = \frac{1}{20}xe^x + \frac{1}{100}e^x}$

Ans $\underline{\underline{y = y_c + y_{p_1} + y_{p_2} = Ae^{3x}\cos 4x + Be^{3x}\sin 4x - \frac{1}{30}\sin 5x + \frac{1}{20}xe^x + \frac{1}{100}e^x}}$

$\boxed{5}$. $y''' - 3y'' + 4y = 0$ $\qquad$ $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$

aux eqn $\quad m^3 - 3m^2 + 4 = 0$

$\boxed{m = -1 \text{ is a root}}$ $\quad (-1-3+4=0) \Rightarrow m+1$ is a factor

$\Rightarrow 0 = m^3 - 3m^2 + 4 = (m+1)(m^2 - 4m + 4) = (m+1)(m-2)^2$

$m_1 = -1 \qquad m_2 = m_3 = 2$

<u>general soln</u> $\quad \boxed{y = Ae^{-x} + Be^{2x} + Cxe^{2x}}$

differentiate to solve for initial conditions:

$$y = Ae^{-x} + Be^{2x} + Cxe^{2x}$$
$$y' = -Ae^{-x} + 2Be^{2x} + C(e^{2x} + 2xe^{2x})$$
$$y'' = Ae^{-x} + 4Be^{2x} + C(4e^{2x} + 4xe^{2x})$$

<u>in'l conditions</u>
$$\begin{cases} A + B = 1 \\ -A + 2B + C = 0 \\ A + 4B + 4C = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} A + B = 1 \\ 3B + C = 1 \\ 3B + 4C = -1 \end{cases} \Longleftrightarrow \begin{cases} A + B = 1 \\ 3B + C = 1 \\ 3C = -2 \end{cases}$$

$\boxed{C = -\tfrac{2}{3}} \quad \boxed{B = \tfrac{5}{9}} \quad \boxed{A = \tfrac{4}{9}}$

soln $\quad \boxed{y = \frac{4}{9}e^{-x} + \frac{5}{9}e^{2x} - \frac{2}{3}xe^{2x}}$

6 (a) $y'' + 5x^{-1}y' + 4x^{-2}y = 0$ (N.B. this is a Cauchy-Euler DE in disguise)

$y_1 = x^{-2}$ is a solution, so substitute

& combine $\cdot (4x^{-2})$ $\cdot (5x^{-1})$ $\cdot (1)$

$$\begin{cases} y = y_1 u = x^{-2}u \\ y' = -2x^{-3}u + x^{-2}u' \\ y'' = +6x^{-4}u - 4x^{-3}u' + x^{-2}u'' \end{cases}$$

so $0 = y'' + 5x^{-1}y' + 4x^{-2}y = (6x^{-4} - 10x^{-4} + 4x^{-4})u + \underbrace{(-4x^{-3} + 5x^{-3})}_{x^{-3}}u' + x^{-2}u''$

double check $= 0$. Not a surprise, since y_1 is a soln

so $\boxed{x^{-3}u' + x^{-2}u'' = 0}$, or equivalently $\boxed{u'' = -x^{-1}u'}$.

substitute $w = u'$, so $\frac{dw}{dx} = u'' = -x^{-1}w$, so $\frac{dw}{w} = -\frac{dx}{x}$, so

$\ln w = -\ln x$ (we only need ONE y_2, otherwise we would have $\ln|w| = -\ln x + C$)

so $\boxed{w = \frac{1}{x}}$ $\therefore$ $u' = \frac{1}{x}$ so we take $\boxed{u = \ln x}$ (we only need ONE u!

the most general u is $u = A\ln x + B$. Check it.)

So $\boxed{y_2 = y_1 u = x^{-2}\ln x}$ is a 2nd solution.

(b) either check if $\forall x$ $C_1x^{-2} + C_2x^{-2}\ln x = 0 \implies$ when $x=1$, $C_1 + C_2 \cdot 0 = 0$ so $C_1 = 0$ $\implies \forall x$, $C_2x^{-2}\ln x = 0$ but then (taking $x = e$) we get $C_2 = 0$ also,

or calculate $W(y_1, y_2) = \det\begin{pmatrix} x^{-2} & x^{-2}\ln x \\ -2x^{-3} & -2x^{-3}\ln x + x^{-2}\cdot\frac{1}{x} \end{pmatrix}$

$= -2x^{-5}\ln x + x^{-5} + 2x^{-5}\ln x = x^{-5}$, not identically zero (in fact, never 0 on a good interval)

$\therefore \{y_1, y_2\}$ are linearly independent.

(c) Look for $y_p = y_1u_1 + y_2u_2 = x^{-2}u_1 + x^{-2}\ln x\, u_2$, where u_1 & u_2 satisfy

$$\begin{cases} x^{-2}u_1' + x^{-2}\ln x\, u_2' = 0 \\ -2x^{-3}u_1' + (-2x^{-3}\ln x + x^{-3})u_2' = x^{-1} \end{cases}$$] the DE is already normalized

$\cdot(x^{+2})$ $\cdot(x^{+3})$ $\iff$ $$\begin{cases} u_1' + \ln x \cdot u_2' = 0 \\ -2u_1' + (-2\ln x + 1)u_2' = x^2 \end{cases}$$ $\boxed{+2}$ G. elim

$\iff \begin{cases} u_1' + \ln x \cdot u_2' = 0 \\ u_2' = x^2 \end{cases}$ $\iff \begin{cases} u_1' = -x^2\ln x \\ u_2' = x^2 \end{cases}$. Hence $\boxed{u_2 = \frac{1}{3}x^3}$, and

$u_1 = \int -x^{+2}\ln x\,dx = -\frac{1}{3}\int \ln x \cdot d(x^3) = -\frac{1}{3}\left(x^3\ln x - \int x^3\frac{dx}{x}\right) = -\frac{1}{3}\left(x^3\ln x - \frac{x^3}{3}\right)$

so $\boxed{u_1 = -\frac{x^3\ln x}{3} + \frac{x^3}{9}}$

hence $\boxed{y_p = x^{-2}\left(-\frac{x^3\ln x}{3} + \frac{x^3}{9}\right) + x^{-2}\left(\frac{1}{3}x^3\right) = \frac{x}{9}}$ (easy to check that this solves $y'' + 5x^{-1}y' + 4x^{-2}y = \frac{x}{9}$)