Math 202 — Spring 2005
Differential Equations, sections 1–4
Quiz 2, April 14 — Duration: 70 minutes

GRADES:

1 (12 pts)	2 (12 pts)	3 (18 pts)	4 (12 pts)	5 (18 pts)	TOTAL/72

YOUR NAME:

YOUR AUB ID#:

Solution by Dr Makdisi:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation F 1	Recitation F 2	Recitation F 12	Recitation F 3
Ms. Zantout	Ms. Zantout	Ms. Zantout	Dr. Yamani

INSTRUCTIONS:

1. Write your NAME and AUB ID number above, and circle your SECTION.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 or 18 points.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Do as much of the exam as you can, and budget your time carefully. If you cannot do a certain integral, just leave it as an integral in your solution for partial credit on the rest of the problem.
5. No calculators, books, or notes allowed. Turn off and put away any cell phones or beepers.

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6. Problems 3 and 5 ask for series solutions. For full credit on these problems, you need to give a formula for the coefficients. HOWEVER, you can get ALMOST FULL CREDIT if you give the first FOUR nonzero terms for any solution. Example: if you were solving the very simple equation $y'' + y = 0$, your solution could just say
$$y = c_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + c_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$
for almost full credit (16/18 points), but for full credit you would have to say that $c_{2\ell} = \frac{(-1)^\ell}{(2\ell)!}c_0$, and that $c_{2\ell+1} = \frac{(-1)^\ell}{(2\ell+1)!}c_1$.

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GOOD LUCK!

An overview of the exam problems.
The problems are repeated inside the booklet — PLEASE
SOLVE EACH PROBLEM ON ITS CORRESPONDING PAGE INSIDE.

Remember to READ the instructions on the front of this exam regarding problems 3 and 5, which ask for series solutions.

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1. (12 points) Find the general solution of the equation

$$y''' + y' - 10y = \cos x + e^{2x}.$$

(Note that y''' is the THIRD derivative of y — do not make the mistake of copying it as y''.)

2. (12 points) Find the general solution of the equation

$$x^2y'' - 2xy' + 2y = \frac{x^3}{\sqrt{1-x^2}}.$$

Note: find y_P using variation of parameters by DIRECTLY substituting $y_P = y_1u_1 + y_2u_2$ into the differential equation. Do NOT use any formulas for variation of parameters unless you prove them first.

3. (18 points) Use a series centered at the point $x = 0$ (which is an ordinary point) to find the general solution of

$$(1 + x^3)y'' - 6xy = 0.$$

(Hint: simplify the recurrence as much as possible before solving for $c_0, c_1, c_2, c_3, \ldots, c_k, \ldots$.)

4. (12 points total, 6 points for each part)
a) Calculate the Wronskian of the FIVE functions

$$y_1 = 1, \qquad y_2 = x, \qquad y_3 = x^2, \qquad y_4 = e^x, \qquad y_5 = e^{2x}.$$

Use this to DEDUCE whether $\{y_1, y_2, y_3, y_4, y_5\}$ are linearly dependent or independent. (Remark: don't be intimidated by the 5×5 determinant. You can evaluate it by expanding by minors along any row or column. Choose a row or column that makes the calculations very easy.)

b) (UNRELATED) Use the SUBSTITUTION $x = e^t$ to solve the following differential equation in terms of Bessel functions. Do NOT write down any series. You MUST carefully explain HOW you do the substitution.

$$\frac{d^2y}{dt^2} + \left(e^{2t} - \frac{1}{25}\right)y = 0.$$

(Recall that the Bessel differential equation is $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, and that its general solution is $y = AJ_\nu(x) + BJ_{-\nu}(x)$ if ν is not an integer, and $y = AJ_\nu(x) + BY_\nu(x)$ if ν is an integer.)

5. (18 points) Given the following differential equation with a regular singular point at $x = 0$:

$$x^2y'' + xy' + (x^2 - 9)y = 0.$$

(This is actually a Bessel differential equation, but we want to find a SERIES solution using the method of Section 6.2.)

Find ONLY ONE solution $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ of this differential equation, corresponding to the LARGER one of the two indicial roots r_1. (Here $r_1 > r_2$ and you should NOT try to find the solution y_2 that uses r_2, since y_2 involves a logarithm.)

1. (12 points) Find the general solution of the equation

$$y''' + y' - 10y = \cos x + e^{2x}.$$

(Note that y''' is the THIRD derivative of y — do not make the mistake of copying it as y''.)

Step I

Find y_c: constant coefficient DE so the auxiliary equation is

$m^3 + m - 10 = 0$. Try roots $m = \pm1, \pm2, \pm5, \pm10$ [factors $(m\pm1)$ or $(m\pm2)$ or $(m\pm5)$ or $(m\pm10)$]

$2^3+2-10=0$

$\therefore m=2$ is a root $\Rightarrow m-2$ is a factor

$$\begin{array}{r|l} & m^2+2m+5 \\ m-2 & m^3 + \quad m - 10 \\ & \ominus\, m^3 \oplus 2m^2 \\ \hline & 2m^2 + m - 10 \\ & \ominus\, 2m^2 \oplus 4m \\ \hline & 5m - 10 \\ & \ominus\, 5m \oplus 10 \\ \hline & 0 \end{array}$$

$\therefore \quad m^3+m-10 = (m-2)(m^2+2m+5)$

moreover the roots of $m^2+2m+5=0$ are $m = -1 \pm \sqrt{-4} = -1\pm 2i$ by the quadratic formula.

Thus $m_1 = 2$, $m_2 = -1+2i$, $m_3 = -1-2i$ roots of aux. equation

$$\boxed{y_c = Ae^{2x} + Be^{-x}\cos 2x + Ce^{-x}\sin 2x}$$

Step II Find y_p: put $y_p = y_{p_1} + y_{p_2}$

we want $\begin{cases} L[y_{p_1}] = \cos x \\ L[y_{p_2}] = e^{2x} \end{cases}$.

for y_{p_1}: we expect y_{p_1} to be a linear combination of $\cos x$ & $\sin x$

so we normally calculate $L[\cos x]$ & $L[\sin x]$

f	f'	f''	f'''	$L[f] = f''' + f' - 10f$
$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\sin x - \sin x - 10\cos x = -10\cos x$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$-\cos x + \cos x - 10\sin x = -10\sin x$

$\longrightarrow$ NOTICE $L[\cos x] = -10\cos x$ so we DONT NEED $\sin x$!

$$\boxed{\text{so } y_{p_1} = -\tfrac{1}{10}\cos x \text{ satisfies } L[y_{p_1}] = \cos x}.$$

for y_{p_2}: we want an output of e^{2x} but 2 is a root of the aux. equation with multiplicity 1. So our input needs an extra factor of x:

$L[xe^{2x}] = (12e^{2x} + 8xe^{2x}) + (e^{2x} + 2xe^{2x}) - 10xe^{2x}$

$= 13e^{2x}$

since here

f	f'	f''	f'''
xe^{2x}	$e^{2x} + 2xe^{2x}$	$2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$	$8e^{2x} + 4e^{2x} + 8xe^{2x} = 12e^{2x} + 8xe^{2x}$

so $\boxed{y_{p_2} = \tfrac{1}{13}xe^{2x}}$

Step III SOLUTION $y = y_c + y_p = Ae^{2x} + Be^{-x}\cos 2x + Ce^{-x}\sin 2x - \frac{1}{10}\cos x + \frac{1}{13}xe^{2x}$

2. (12 points) Find the general solution of the equation

$$x^2y'' - 2xy' + 2y = \frac{x^3}{\sqrt{1-x^2}}.$$

Note: find y_P using variation of parameters by DIRECTLY substituting $y_P = y_1u_1 + y_2u_2$ into the differential equation. Do NOT use any formulas for variation of parameters unless you prove them first.

Step I y_c. This is Cauchy-Euler so we subst. $y = x^r$ in $x^2y'' - 2xy' + 2y = 0$ (note $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$).

we get $r(r-1)x^{r-2+2} - 2rx^{r-1+1} + 2x^r = 0$. Cancelling x^r, we g

auxiliary equation $r(r-1) - 2r + 2 = 0$

$\Leftrightarrow r^2 - 3r + 2 = 0$

$\Leftrightarrow (r-1)(r-2) = 0$

roots $r_1 = 1 \Rightarrow y_1 = x$
$r_2 = 2 \Rightarrow y_2 = x^2$ solns of homog equation $\therefore \boxed{y_c = Ax + Bx^2}$

Step II look for $y_p = xu_1 + x^2u_2$ ①

$\Rightarrow y_p' = u_1 + 2xu_2 + \underbrace{xu_1' + x^2u_2'}_{0}$ ② we CHOOSE $\boxed{xu_1' + x^2u_2' = 0}$ to avoid seeing u_1'', u_2'' la

$\Rightarrow y_p'' = + 2u_2 + u_1' + 2xu_2'$ ③

take $2\cdot$① $- 2x\cdot$② $+ x^2\cdot$③ to get

$$\frac{x^3}{\sqrt{1-x^2}} = 2y_p - 2xy_p' + x^2y_p'' = 0u_1 + 0u_2 + x^2u_1' + 2x^3u_2'$$ (check mentally that they're 0)

copy $\boxed{x^2u_1' + 2x^3u_2' = \frac{x^3}{\sqrt{1-x^2}}}$

So we must solve the SYSTEM $\begin{cases} xu_1' + x^2u_2' = 0 \\ x^2u_1' + 2x^3u_2' = \frac{x^3}{\sqrt{1-x^2}} \end{cases} \overset{\div x}{\underset{\div x^2}{\Longleftrightarrow}} \begin{cases} u_1' + xu_2' = 0 \\ u_1' + 2xu_2' = \frac{x}{\sqrt{1-}} \end{cases}$

from Ⓑ−Ⓐ we get $xu_2' = \frac{x}{\sqrt{1-x^2}} \Rightarrow u_2' = \frac{1}{\sqrt{1-x^2}} \overset{\text{integrate}}{\Longrightarrow} \boxed{u_2 = \sin^{-1}x}$

from 2Ⓐ−Ⓑ we get $u_1' = \frac{-x}{\sqrt{1-x^2}} \Rightarrow u_1 = \int\frac{-x\,dx}{\sqrt{1-x^2}} = \int\frac{d(1-x^2)}{2\sqrt{1-x^2}} = \boxed{\sqrt{1-x^2}}$

Step III The general solution is $y = y_c + y_p = Ay_1 + By_2 + y_1u_1 + y_2u_2$

$$\boxed{\therefore\ y = Ax + Bx^2 + x\sqrt{1-x^2} + x^2\sin^{-1}x}$$

3. (18 points) Use a series centered at the point $x = 0$ (which is an ordinary point) to find the general solution of

$$(1 + x^3)y'' - 6xy = 0.$$

(Hint: simplify the recurrence as much as possible before solving for $c_0, c_1, c_2, c_3, \ldots, c_k, \ldots$)

Since we know this is an ordinary point, we use $y = \sum_{n=0}^{\infty} c_n x^n$ (no r)

then as usual $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$

$y'' + x^3 y'' - 6xy = 0 \iff \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty} n(n-1) c_n x^{n+1} - \sum_{n=0}^{\infty} 6c_n x^{n+1}}_{\text{combine}} = 0$

using $n(n-1) - 6 = n^2 - n - 6 \overset{\text{factor}}{=} (n-3)(n+2)$

$\iff \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} (n-3)(n+2) c_n x^{n+1} = 0$

reindex $n-2 = k+1$ so $n = k+3$, $n=0 \to k=-3$

reindex $k=n$ so $n+1=k+1$

$\iff \sum_{k=-3}^{\infty} (k+3)(k+2) c_{k+3} x^{k+1} + \sum_{k=0}^{\infty} (k-3)(k+2) c_k x^{k+1} = 0$

note same power x^{k+1}

Remark ANOTHER WAY is to keep the 1 sum with $k=n$ (so $n-2=$…) & reindex the second sum to make $n+1 = k-2$ OR you could reindex both sums to have x^k in both places, but this would be more work than necessary.

$\therefore$ Initial identities $\begin{cases} k=-3 \Rightarrow (-3+3)(-3+2)c_0 = 0 \Rightarrow 0c_0 = 0 \\ k=-2 \Rightarrow (-2+3)(-2+2)c_1 = 0 \Rightarrow 0c_1 = 0 \\ k=-1 \Rightarrow (-1+3)(-1+2)c_2 = 0 \Rightarrow 2c_2 = 0 \end{cases}$

$\boxed{\text{Conclusion}}$ c_0 & c_1 are arbitrary but $\boxed{c_2 = 0}$.

Recurrence: for $k \geq 0$ $(k+3)(k+2)c_{k+3} + (k-3)(k+2)c_k = 0$ (note $k+2 \geq 2$ so $k+2 \neq 0$)

$$\Rightarrow \boxed{c_{k+3} = -\frac{(k-3)}{(k+3)} c_k} \qquad \Delta k = 3$$

$c_0 \xrightarrow[(k=0)]{\cdot -\frac{(-3)}{3}} c_3 \xrightarrow[(k=3)]{\cdot \boxed{0}} c_6 = \boxed{0} \xrightarrow{\cdot\text{any}} c_9 = \boxed{0} \rightarrow \cdots \sim \boxed{c_{3l} = 0 \text{ for } 3l \geq 6}$ with $c_3 = +c_0$

$c_1 \xrightarrow[(k=1)]{\cdot -\frac{(-2)}{4}} c_4 \xrightarrow[(k=4)]{\cdot -\frac{(1)}{7}} c_7 \xrightarrow[(k=7)]{\cdot -\frac{(4)}{10}} c_{10} \xrightarrow{\cdot -\frac{(7)}{13}} \cdots c_{3l-2} \xrightarrow[(k=3l-2)]{\cdot -\frac{(3l-5)}{(3l+1)}} c_{3l+1}$

$$\Rightarrow \boxed{c_{3l+1} = (-1)^l \cdot \frac{(-2)(1)(4)\cdots(3l-5)}{(4)(7)(10)\cdots(3l+1)} c_1}$$

CONTINUED P.(

$\boxed{c_2 = 0} \xrightarrow[(k=2)]{\cdot\text{any}} c_5 = \boxed{0} \longrightarrow c_8 = \boxed{0} \longrightarrow \cdots \longrightarrow \boxed{c_{3l+2} = 0}$ for all l

4. (12 points total, 6 points for each part)

a) Calculate the Wronskian of the FIVE functions

$$y_1 = 1, \qquad y_2 = x, \qquad y_3 = x^2, \qquad y_4 = e^x, \qquad y_5 = e^{2x}.$$

Use this to DEDUCE whether $\{y_1, y_2, y_3, y_4, y_5\}$ are linearly dependent or independent. (Remark: don't be intimidated by the 5×5 determinant. You can evaluate it by expanding by minors along any row or column. Choose a row or column that makes the calculations very easy.)

$$W = \det\begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1' & y_2' & y_3' & y_4' & y_5' \\ y_1'' & - & - & - & y_5'' \\ y_1''' & - & - & - & y_5''' \\ y_1'''' & - & - & - & y_5'''' \end{pmatrix} = \det\begin{pmatrix} 1 & x & x^2 & e^x & e^{2x} \\ 0 & 1 & 2x & e^x & 2e^{2x} \\ 0 & 0 & 2 & e^x & 4e^{2x} \\ 0 & 0 & 0 & e^x & 8e^{2x} \\ 0 & 0 & 0 & e^x & 16e^{2x} \end{pmatrix}$$

use 1st column

$$= 1 \cdot \det\begin{pmatrix} 1 & 2x & e^x & 2e^{2x} \\ 0 & 2 & e^x & 4e^{2x} \\ 0 & 0 & e^x & 8e^{2x} \\ 0 & 0 & e^x & 16e^{2x} \end{pmatrix} + 0 \cdot (\text{other stuff}) = 1 \cdot 1 \cdot \det\begin{pmatrix} 2 & e^x & 4e^{2x} \\ 0 & e^x & 8e^{2x} \\ 0 & e^x & 16e^{2x} \end{pmatrix} + 0 \cdot (\text{other stuff})$$

use this column use this column

$$= 1 \cdot 1 \cdot 2 \cdot \det\begin{pmatrix} e^x & 8e^{2x} \\ e^x & 16e^{2x} \end{pmatrix} = 2 \cdot (16e^{3x} - 8e^{3x}) = \boxed{16e^{3x}}.$$

The Wronskian is not identically zero, so $\{y_1, \dots y_5\}$ are linearly INDEPENDENT.

b) (UNRELATED) Use the SUBSTITUTION $x = e^t$ to solve the following differential equation in terms of Bessel functions. Do NOT write down any series. You MUST carefully explain HOW you do the substitution.

$$\frac{d^2y}{dt^2} + \left(e^{2t} - \frac{1}{25}\right)y = 0.$$

(Recall that the Bessel differential equation is $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, and that its general solution is $y = AJ_\nu(x) + BJ_{-\nu}(x)$ if ν is not an integer, and $y = AJ_\nu(x) + BY_\nu(x)$ if ν is an integer.)

here $\dfrac{df}{dt} \underset{\text{CHAIN RULE}}{=} \dfrac{df}{dx} \cdot \dfrac{dx}{dt} = \dfrac{df}{dx} \cdot \dfrac{d(e^t)}{dt} = e^t \cdot \dfrac{df}{dx} = x\dfrac{df}{dx}$

So "$\dfrac{d}{dt} = x\dfrac{d}{dx}$". In particular

$$\boxed{\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) \underset{\text{PRODUCT RULE}}{=} x\left[1 \cdot \frac{dy}{dx} + x \cdot \frac{d^2y}{dx^2}\right] = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}}$$

Substituting in our DE, & using $e^{2t} = (e^t)^2 = x^2$, we obtain

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{25}\right)y = 0.$$ This is Bessel with $\nu = \frac{1}{5}$

The general solution in terms of x is $y = AJ_{\frac{1}{5}}(x) + BJ_{-\frac{1}{5}}(x)$.

We must rewrite this in terms of t:

$$\boxed{y = AJ_{\frac{1}{5}}(e^t) + BJ_{-\frac{1}{5}}(e^t)}$$

5. (18 points) Given the following differential equation with a regular singular point at $x = 0$:

$$x^2y'' + xy' + (x^2 - 9)y = 0.$$

(This is actually a Bessel differential equation, but we want to find a SERIES solution using the method of Section 6.2.)

Find ONLY ONE solution $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ of this differential equation, corresponding to the LARGER one of the two indicial roots r_1. (Here $r_1 > r_2$ and you should NOT try to find the solution y_2 that uses r_2, since y_2 involves a logarithm.)

substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$,

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$ into the DE

$$x^2y'' \quad + \quad xy' \quad + x^2y \quad -9y = 0$$

$$\underline{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}} + \underline{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \underline{\sum_{n=0}^{\infty} 9c_n x^{n+r}} = 0$$

combine terms with same power x^{n+r}

the coefficient is $\left[\underbrace{(n+r)(n+r-1) + (n+r)} - 9\right] c_n = (n+r+3)(n+r-3)c_n$.

put $u = n+r$; this is $u(u-1) + u - 9$

$= u^2 - u + u - 9 = u^2 - 9 = (u+3)(u-3)$

So $$\sum_{n=0}^{\infty} (n+r+3)(n+r-3)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$k = n$ $(k+r = n+r)$

$k = n+2$
$(n+r+2 = k+r)$
so $n = k-2$, $n=0 \Leftrightarrow k = 2$

$$\sum_{k=0}^{\infty}(k+r+3)(k+r-3)c_k x^{k+r} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r} = 0$$

<u>initial identities</u> $\quad k=0 \quad (r+3)(r-3)c_0 = 0 \longrightarrow$ we always take $c_0 \neq 0$ so $\boxed{(r+3)(r-3) = 0}$ indicial equation

$k=1 \quad (r+4)(r-2)c_1 = 0$

<u>recurrence</u> $\quad k \geq 2 \Longrightarrow (k+r+3)(k+r-3)c_k + c_{k-2} = 0$

$$c_k = \frac{-c_{k-2}}{(k+r+3)(k+r-3)}$$

The indicial roots are $\boxed{\begin{array}{l} r_1 = +3 \\ r_2 = -3 \end{array}}$

Continued on p. 7

Continuation of ex 3

the first few coefficients are:

$c_0 = \text{any}, \quad c_3 = c_0, \quad c_6 = 0, \quad c_9 = 0$

$c_1 = \text{any}, \quad c_4 = -\frac{(-2)}{4} c_1, \quad c_7 = +\frac{(-2)(1)}{(4)(7)} c_1, \quad c_{10} = -\frac{(-2)(1)(4)}{(4)(7)(10)} c_1$

$(c_2 = c_5 = c_8 = c_{11} = 0)$

the solution is

$$y = c_0 \left[1 + x^3 \right] + c_1 \left[x - \frac{(-2)}{4} x^4 + \frac{(-2)(1)}{(4)(7)} x^7 - \frac{(-2)(1)(4)}{(4)(7)(10)} x^{10} + \cdots + (-1)^\ell \frac{(-2)(1)(4)\cdots(3\ell-5)}{(4)(7)(10)\cdots(3\ell+1)} x^{3\ell+1} + \cdots \right]$$

note no $+\cdots$ since the coefficients are 0

Remark in fact the coefficient of $x^{3\ell+1}$ can be simplified:

$$(-1)^\ell \frac{(-2)(1)\cancel{(4)}\cancel{(7)}\cancel{(10)}\cdots(3\ell-11)(3\ell-8)(3\ell-5)}{\cancel{(4)}\cancel{(7)}\cancel{(10)}\cdots\cdots(3\ell-11)(3\ell-8)(3\ell-5)(3\ell-2)(3\ell+1)} x^{3\ell+1}$$

$$= (-1)^\ell \cdot \frac{(-2)}{(3\ell-2)(3\ell+1)} x^{3\ell+1} = (-1)^{\ell+1} \cdot \frac{2}{(3\ell-2)(3\ell+1)} x^{3\ell+1}$$

But I wasn't expecting people to notice this.

Ex. 5 continued

We now substitute $r=3$ so

initial identities $\begin{cases} 0c_0 = 0 & \text{(we know that)} \\ (7)(1)\,c_1 = 0 \end{cases}$ $\boxed{\text{THEREFOR } c_1=0}$

recurrence: $c_k = \dfrac{-c_{k-2}}{(k+6)k}$ for $k \geq 2$ $(\Delta k = 2)$

$$c_0 \xrightarrow[(k=2)]{-\frac{1}{8\cdot 2}} c_2 \xrightarrow[(k=4)]{-\frac{1}{10\cdot 4}} c_4 \xrightarrow{-\frac{1}{12\cdot 6}} c_6 \longrightarrow \cdots c_{2l-2} \xrightarrow[(k=2l)]{\frac{-1}{(2l+6)(2l)}} c_{2l}$$

$$c_1 = \boxed{0} \xrightarrow[(k=3)]{\frac{-1}{\text{any}}} c_3 = 0 \longrightarrow c_5 = 0 \longrightarrow \cdots - \longrightarrow \boxed{c_{2l+1} = 0}$$

here $c_2 = \dfrac{-1}{8\cdot 2} c_0$, $c_4 = +\dfrac{1}{(8\cdot 10)(2\cdot 4)} c_0$, $c_6 = -\dfrac{1}{(8\cdot 10\cdot 12)(2\cdot 4\cdot 6)} c_0$,

$$\cdots \quad c_{2l} = \frac{(-1)^l}{[8\cdot 10\cdot 12\cdots(2l+6)][2\cdot 4\cdot 6\cdots(2l)]} c_0$$

$c_1 = c_3 = \cdots = 0$

$$y_1 = c_0 x^{3+0} + c_1 x^{3+1} + c_2 x^{3+2} + \cdots$$

$$= c_0 \left[x^{3+0} - \frac{1}{8\cdot 2} x^{3+1} + \frac{1}{(8\cdot 10)(2\cdot 4)} x^{3+2} - \frac{1}{(8\cdot 10\cdot 12)(2\cdot 4\cdot 6)} x^{3+3} + \cdots \right.$$

$$\left. + \frac{(-1)^l}{[8\cdot 10\cdot 12\cdots(2l+6)][2\cdot 4\cdot 6\cdots(2l)]} x^{2l+3} + \cdots \right.$$

optional (can take $c_0 = 1$ since we want ONE solution)

(Remark we can rewrite the term $\dfrac{(-1)^l}{[8\cdot 10\cdot 12\cdots(2l+6)][2\cdot 4\cdot 6\cdots(2l)]} x^{2l+3}$

as $\dfrac{2\cdot 4\cdot 6\cdot(-1)^l}{2^{2l+3}\,(l+3)!\,l!} \cdot x^{2l+3}$)