Math 202 Quiz 2 Spring 2005-06

Sections 1-4 (Professor Kamal Khuri-Makdisi)

1. (16 points; please try to solve on this page and the next one) Find the general solution of the differential equation

$$y'' + 5y' + 4y = x + 3 + \sin 2x + \frac{1}{e^{2x} - 1}.$$

Hint: part of this problem needs variation of parameters. You are allowed to write down the system of equations involving u_1 and u_2 directly, without deriving them. For the integrations, I suggest using the substitution $w = e^x$ and some partial fractions.

y_c: auxiliary eqn $m^2 + 5m + 4 = 0$ roots $-1, -4$ $\boxed{y_c = Ae^{-x} + Be^{-4x}}$

$(m+1)(m+4) = 0$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$ where $L[y_{p_1}] = x+3$, $L[y_{p_2}] = \sin 2x$, $L[y_{p_3}] = \frac{1}{e^{2x}-1}$.

$\boxed{y_{p_1}}$: $L[x] = 5 + 4x$ · $L[Ax+B] = A(5+4x) + B(4) = (5A+4B) + 4Ax \overset{\text{want}}{=} x+3$

$L[1] = 4$

$\begin{cases} 5A+4B = 3 \\ 4A = 1 \end{cases} \iff \begin{cases} A = 1/4 \\ B = \frac{3 - 5/4}{4} = \frac{7}{16} \end{cases}$ $\boxed{y_{p_1} = \frac{1}{4}x + \frac{7}{16}}$ (note $\left.\begin{matrix} L[\frac{1}{4}] = 1 \\ L[\frac{1}{4}x - \frac{5}{16}] = x \\ \& \ y_{p_1} = \frac{1}{4}x - \frac{5}{16} + \frac{3}{4} \end{matrix}\right\}$ but this takes longer)

$\boxed{y_{p_2}}$: $L[\cos 2x] = -10 \sin 2x$ so $L\left[-\frac{1}{10}\cos 2x\right] = \sin 2x$. $\boxed{y_{p_2} = -\frac{1}{10}\cos 2x}$

$(L[\sin 2x] = 10 \cos 2x)$

$\boxed{y_{p_3}}$: put $y_{p_3} = u_1 y_1 + u_2 y_2$ $\begin{pmatrix} y_1 = e^{-x} \\ y_2 = e^{-4x} \end{pmatrix}$ where $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \frac{1}{e^{2x}-1} \end{cases}$ (DE is already normalized)

$\iff \begin{cases} e^{-x} u_1' + e^{-4x} u_2' = 0 & ① \\ -e^{-x} u_1' - 4e^{-4x} u_2' = \frac{1}{e^{2x}-1} & ② \end{cases}$ $\iff \begin{cases} e^{-x} u_1' + e^{-4x} u_2' = 0 & ① \\ -3e^{-4x} u_2' = \frac{1}{e^{2x}-1} & ②+① \end{cases} \implies u_2' = -\frac{1}{3} \frac{e^{4x}}{e^{2x}-1}$

$\& ① \implies u_1' = +\frac{1}{3} \frac{e^x}{e^{2x}-1}$

$u_1 = \frac{1}{3}\int \frac{e^x\,dx}{e^{2x}-1} = \frac{1}{3}\int \frac{dw}{w^2-1} \overset{\text{where } w = e^x}{=} \frac{1}{3}\int\left[\frac{1/2}{w-1} + \frac{-1/2}{w+1}\right]dw = \frac{1}{6}\ln|w-1| - \frac{1}{6}\ln|w+1|$

$= \frac{1}{6}\ln|e^x - 1| - \frac{1}{6}\ln|e^x+1|$

$u_2 = -\frac{1}{3}\int \frac{\overbrace{e^{4x}\,dx}^{e^{3x}d(e^x)}}{e^{2x}-1} = -\frac{1}{3}\int \frac{w^3\,dw}{w^2-1}$ divide $w^2-1 \overset{w}{\overline{\big)\, w^3}}$, $\frac{-(w^3 - w)}{w}$ $\therefore \frac{w^3}{w^2-1} = w + \frac{w}{w^2-1}$

$= -\frac{1}{3}\int\left(w + \frac{w}{w^2-1}\right)dw = -\frac{1}{3}\left[\frac{w^2}{2} + \frac{1}{2}\ln|w^2-1|\right] = -\frac{w^2}{6} - \frac{\ln|w^2-1|}{6} = -\frac{e^{2x}}{6} - \frac{\ln|e^{2x}-1|}{6}$

note $\frac{w\,dw}{w^2-1} = \frac{1}{2}\frac{d(w^2-1)}{w^2-1}$

$\boxed{y_{p_3} = e^{-x}\left[\frac{1}{6}\ln|e^x-1| - \frac{1}{6}\ln|e^x+1|\right] + e^{-4x}\left[-\frac{e^{2x}}{6} - \frac{\ln|e^{2x}-1|}{6}\right]}$

$\boxed{y = y_c + y_{p_1} + y_{p_2} + y_{p_3} = Ae^{-x} + Be^{-4x} + \frac{1}{4}x + \frac{7}{16} - \frac{1}{10}\cos 2x + \frac{e^{-x}}{6}\left(\ln\left|\frac{e^x-1}{e^x+1}\right|\right) - \frac{e^{-4x}}{6}\left(e^{2x} + \ln|e^{2x}-1|\right)}$

2. (16 points; please try to solve on this page and the next one) Using a series centered at $x = 0$, find the general solution of

$$y'' - 3xy' - y = 0.$$

$x=0$ is an ordinary pt, so let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$.
$(P = -3x,\ Q = -1)$

DE becomes:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty} (-3) n c_n x^n + \sum_{n=0}^{\infty} (-1) c_n x^n}_{\text{combine}} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} (3n+1) c_n x^n = 0$$

copy $k = n$; $n = k-2$, $n=0 \Leftrightarrow k=2$ (shift) (*) see below

$$\sum_{k=0}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=2}^{\infty} \underbrace{(3(k-2)+1)}_{(3k-5)} c_{k-2} x^{k-2} = 0$$

startup eqns
$k=0 \Rightarrow 0 c_0 = 0$
$k=1 \Rightarrow 0 c_1 = 0$ } c_0, c_1 arbitrary

recurrence
$k \geq 2 \Rightarrow k(k-1) c_k - (3k-5) c_{k-2} = 0$

$$\boxed{c_k = \frac{3k-5}{k(k-1)} c_{k-2}}$$ (2 term recurrence)

$$c_0 \xrightarrow[(k=2)]{\frac{1}{2\cdot1}} c_2 \xrightarrow[(k=4)]{\frac{7}{4\cdot3}} c_4 \xrightarrow[(k=6)]{\frac{13}{6\cdot5}} c_6 \longrightarrow \cdots \longrightarrow c_{2\ell-2} \xrightarrow[k=2\ell]{\frac{6\ell-5}{2\ell(2\ell-1)}} c_{2\ell}$$ after ℓ hops

$c_2 = \frac{1}{2!} c_0$, $c_4 = \frac{1\cdot7}{4!} c_0$, $c_6 = \frac{1\cdot7\cdot13}{6!} c_0$, $\ldots$, $\boxed{c_{2\ell} = \frac{1\cdot7\cdot13\cdots(6\ell-5)}{(2\ell)!} c_0 = \frac{\prod_{j=1}^{\ell}(6j-5)}{(2\ell)!} c_0}$ ...

$$c_1 \xrightarrow[(k=3)]{\frac{4}{3\cdot2}} c_3 \xrightarrow[(k=5)]{\frac{10}{5\cdot4}} c_5 \xrightarrow[(k=7)]{\frac{16}{7\cdot6}} c_7 \longrightarrow \cdots \longrightarrow c_{2\ell-1} \xrightarrow[(k=2\ell+1)]{\frac{6\ell-2}{(2\ell+1)(2\ell)}} c_{2\ell+1}$$ after ℓ hops

$c_3 = \frac{4}{3!} c_1$, $c_5 = \frac{4\cdot10}{5!} c_1$, $c_7 = \frac{4\cdot10\cdot16}{7!} c_1$, $\ldots$, $\boxed{c_{2\ell+1} = \frac{4\cdot10\cdot16\cdots(6\ell-2)}{(2\ell+1)!} c_1 = \frac{\prod_{j=1}^{\ell}(6j-2)}{(2\ell+1)!} c_1}$

soln. $y = c_0 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \cdots$
$\quad + c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \cdots$

$$= c_0 \left[1 + \frac{1}{2!} x^2 + \frac{1\cdot7}{4!} x^4 + \frac{1\cdot7\cdot13}{6!} x^6 + \cdots + \frac{1\cdot7\cdot13\cdots(6\ell-5)}{(2\ell)!} x^{2\ell} + \cdots \right]$$

$$+ c_1 \left[x + \frac{4}{3!} x^3 + \frac{4\cdot10}{5!} x^5 + \frac{4\cdot10\cdot16}{7!} x^7 + \cdots + \frac{4\cdot10\cdot16\cdots(6\ell-2)}{(2\ell+1)!} x^{2\ell+1} + \cdots \right]$$

you can also write
$$\boxed{y = c_0 \sum_{\ell=0}^{\infty} \frac{\prod_{j=1}^{\ell}(6j-5)}{(2\ell)!} x^{2\ell} + c_1 \sum_{\ell=0}^{\infty} \frac{\prod_{j=1}^{\ell}(6j-2)}{(2\ell+1)!} x^{2\ell+1}}$$

(*) Remark you can also shift the 1st & copy the 2nd to get

$$\sum_{k=-2}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=0}^{\infty} (3k+1) c_k x^k = 0 \Longrightarrow$$ same startup eqns, & recurrence is $c_{k+2} = \frac{3k+1}{(k+2)(k+1)} c_k$ for $k \geq 0$

(of course this leads to the same solution)

3. (16 points; please try to solve on this page and the next one) Using a **series centered** at $x = 0$, find the general solution of

$$2x^2y'' + (x^2 - 3x)y' + 3y = 0.$$

Here $P = \frac{x^2-3x}{2x^2} = \frac{x-3}{2x}$, $Q = \frac{3}{2x^2}$ so $x=0$ is a regular singular pt.

x in denom $\qquad$ x^2 in denom

we use $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$.

Substitute in $2x^2y'' + x^2y' - 3xy' + 3y = 0$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty}\left(-3(n+r)\right)c_n x^{n+r} + \sum_{n=0}^{\infty} 3c_n x^{n+r} = 0$$

combine underlined terms & factor

$$\sum_{n=0}^{\infty} \left[(2(n+r)-3)(n+r-1)\right] c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1} = 0$$

$n=k$ (copy) $\qquad$ $n+1=k$ (shift), $n=k-1$, $n=0 \Leftrightarrow k=1$

(Scratch for [ ]: if $v=n+r$, we have a coeff. $2v(v-1)-3v+3 = 2v^2-5v+3 = (2v-3)(v-1)$)

$$\sum_{k=0}^{\infty} \left(2(k+r)-3\right)(k+r-1)c_k x^{k+r} + \sum_{k=1}^{\infty} (k+r-1)c_{k-1}x^{k+r} = 0$$

Starting equation: $k=0$, $(2r-3)(r-1)c_0 = 0$

BUT $c_0 \neq 0$ so $\boxed{r = \frac{3}{2} \text{ or } r=1}$

(note the roots differ by $\frac{3}{2}-1 = \frac{1}{2}$, NOT an integer, so we will get 2 series solutions)

recurrence

$k \geq 1 \Rightarrow (2(k+r)-3)(k+r-1)c_k + (k+r-1)c_{k-1} = 0$

note that $k\geq 1$, $r=\frac{3}{2}$ or $r=1 \Rightarrow r\geq 1$ $\Rightarrow k+r-1 \geq 1+1-1 > 0$

so $k+r-1 \neq 0$ & we can cancel it from now, if we want. However, I will cancel it later instead.

$\boxed{\text{1st solution } y_1}$, $r=\frac{3}{2}$, recurrence says $\left(2(k+\frac{3}{2})-3\right)(k+\frac{3}{2}-1)c_k + (k+\frac{3}{2}-1)c_{k-1} = 0$ for $k\geq 1$

$\Leftrightarrow 2k\cancel{(k+\frac{1}{2})}c_k + \cancel{(k+\frac{1}{2})}c_{k-1} = 0$ $\qquad$ but $k\geq 1 \Rightarrow k+\frac{1}{2}\neq 0$ (this is $k+r-1$ which I've decided to cancel here)

so cancel & get $\boxed{c_k = \frac{-1}{2k}c_{k-1}}$ for $k\geq 1$

$c_0 \xrightarrow[{[k=1]}]{\cdot -\frac{1}{2}} c_1 \xrightarrow[{[k=2]}]{\cdot -\frac{1}{4}} c_2 \xrightarrow[{[k=3]}]{\cdot -\frac{1}{6}} c_3 \longrightarrow \cdots \longrightarrow c_{\ell-1} \xrightarrow[{[k=\ell]}]{\frac{-1}{2\ell}} c_\ell$, $\quad c_1 = -\frac{1}{2}c_0$, $c_2 = +\frac{1}{2\cdot 4}c_0$, $c_3 = \frac{-1}{2\cdot 4\cdot 6}c_0$ $\quad \boxed{c_\ell = \frac{(-1)^\ell}{2\cdot 4\cdot 6\cdots(2\ell)}c_0}$ (note $c_\ell = \frac{(-1)^\ell}{2^\ell \ell!}c_0$)

got solution $y_1 = c_0x^{3/2} + c_1x^{1+3/2} + c_2x^{2+3/2} + \cdots$

$$= c_0\left[x^{3/2} - \frac{1}{2}x^{1+3/2} + \frac{1}{2\cdot 4}x^{2+3/2} - \frac{1}{2\cdot 4\cdot 6}x^{3+3/2} + \cdots + \frac{(-1)^\ell}{\prod_{j=1}^{\ell}(2j)}x^{\ell+3/2} + \cdots\right]$$

(can take $c_0=1$)

note we can rewrite $y_1 = c_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2^\ell \ell!}x^{\ell+3/2}$ (take $c_0=1$) $\quad$ (by the way, note $y_1 = x^{3/2}e^{-x/2}$)

3 — continued.

$\boxed{\text{2nd solution } y_2}$ $r=1$, recurrence says $(2(k+1)-3)(k+1-1)c_k + (k+1-1)\,c_{k-1} = 0$ for $k \geq 1$

$\Leftrightarrow (2k-1)\cancel{k}\,c_k + \cancel{k}\,c_{k-1} = 0$ for $k\geq 1$ i.e. $k \neq 0$ so it can be cancelled

$$\Leftrightarrow \boxed{c_k = -\frac{c_{k-1}}{2k-1}}$$

$$c_0 \xrightarrow[{[k=1]}]{-\frac{1}{1}} c_1 \xrightarrow[{[k=2]}]{-\frac{1}{3}} c_2 \xrightarrow[{[k=3]}]{-\frac{1}{5}} c_3 \longrightarrow \cdots \quad c_{\ell-1} \xrightarrow[{[k=\ell]}]{\frac{-1}{2\ell-1}} c_\ell$$

$c_1 = -\frac{1}{1}c_0$, $c_2 = +\frac{1}{1\cdot 3}c_0$, $c_3 = \frac{-1}{1\cdot 3\cdot 5}c_0$, $\cdots$ $\quad c_\ell = \frac{(-1)^\ell}{1\cdot 3\cdot 5\cdots(2\ell-1)}\,c_0$

i.e. $\boxed{c_\ell = \frac{(-1)^\ell}{\prod_{j=1}^{\ell}(2j-1)}\,c_0}$ (exercise: check that $c_\ell = \frac{(-2)^\ell \ell!}{(2\ell)!}c_0$)

soln. $y_2 = c_0x^1 + c_1x^{1+1} + c_2x^{2+1} + c_3x^{3+1} + \cdots$

$$= \cancel{c_0}\left[x^1 - \frac{1}{1}x^{1+1} + \frac{1}{1\cdot 3}x^{2+1} - \frac{1}{1\cdot 3\cdot 5}x^{3+1} + \cdots + \frac{(-1)^\ell}{1\cdot 3\cdot 5\cdots(2\ell-1)}x^{\ell+1} + \cdots\right]$$

can take $c_0 = 1$

$$\text{so } y_2 = \cancel{c_0}\sum_{\ell=0}^{\infty}\frac{(-2)^\ell \ell!}{(2\ell)!}x^{\ell+1}$$

<u>SOLUTION</u> $y = Ay_1 + By_2$

$$= \boxed{A\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2^\ell \ell!}x^{\ell+3/2} + B\sum_{\ell=0}^{\infty}\frac{(-2)^\ell \ell!}{(2\ell)!}x^{\ell+1}}$$

4. (8 points) Using any method that you like, find the general solution of

$$5x^2y'' + 3xy' + y = 0.$$

This is a Cauchy-Euler equation.

Trying a solution of the form $y = x^r$, we get

$5r(r-1)\cancel{x^r} + 3r\cancel{x^r} + 1 \cdot x^r = 0$ so $\boxed{5r(r-1)+3r+1=0}$ auxiliary equation.

$\iff 5r^2 - 2r + 1 = 0$ so $r = \dfrac{+2 \pm \sqrt{2^2 - 4 \cdot 5 \cdot 1}}{2 \cdot 5} = \dfrac{1 \pm 2i}{5}$ solutions by quadratic eqn

so we have complex solutions $y_1 = x^{\frac{1}{5} + \frac{2}{5}i} = x^{\frac{1}{5}}\left(e^{i(\frac{2}{5}\ln x)}\right) = x^{\frac{1}{5}}\left(\cos\left(\frac{2}{5}\ln x\right) + i\sin\left(\frac{2}{5}\ln x\right)\right)$

$y_2 = x^{\frac{1}{5} - \frac{2}{5}i} = \text{- - - -} = x^{\frac{1}{5}}\left(\cos\left(\frac{2}{5}\ln x\right) - i\sin\left(\frac{2}{5}\ln x\right)\right)$

& real solns $z_1 = \dfrac{y_1 + y_2}{2} = x^{\frac{1}{5}}\cos\left(\frac{2}{5}\ln x\right)$

$z_2 = \dfrac{y_1 - y_2}{2i} = x^{\frac{1}{5}}\sin\left(\frac{2}{5}\ln x\right)$

for a general soln $\boxed{y = Ax^{\frac{1}{5}}\cos\left(\frac{2}{5}\ln x\right) + Bx^{\frac{1}{5}}\sin\left(\frac{2}{5}\ln x\right)}$.

[2nd method: make a substitution $x = e^t$ (details left to you)

to get $5\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = 0$ (same aux eqn but now constant coeff)

with roots $m_1, m_2 = \dfrac{1 \pm 2i}{5}$ of the auxiliary equation

$\implies y = Ae^{\frac{1}{5}t}\cos\left(\frac{2}{5}t\right) + Be^{\frac{1}{5}t}\sin\left(\frac{2}{5}t\right)$

$= Ax^{\frac{1}{5}}\cos\left(\frac{2}{5}\ln x\right) + Bx^{\frac{1}{5}}\sin\left(\frac{2}{5}\ln x\right)$]

5. (8 points)

a) Use the Wronskian determinant to show that the functions $y_1 = x$, $y_2 = \cos x$, and $y_3 = \sin x$ are linearly independent.

$$W(y_1, y_2, y_3) = \det\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \det\begin{pmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{pmatrix}$$

expand along 1st column (any other choice works just as well, but this is easiest)

$$= x \cdot \det\begin{pmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{pmatrix} - 1 \cdot \det\begin{pmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{pmatrix} + 0 \cdot \det(\text{don't care})$$

$$= x \cdot \left[+\sin^2 x - (-\cos^2 x)\right] - 1 \cdot \underbrace{\left[\begin{matrix} -\cos x \sin x \\ +\cos x \sin x \end{matrix}\right]}_{0}$$

$$= x\,(\sin^2 x + \cos^2 x) = x,$$

Since $W(y_1, y_2, y_3)$ is not identically zero, we conclude that y_1, y_2, and y_3 are linearly independent.

[Ex find a 3rd order homogeneous linear DE whose general solution is $y = Ax + B\cos x + C\sin x$]

b) WITHOUT using the Wronskian determinant, show directly from the definition that the functions $y_1 = 1$, $y_2 = e^x$, and $y_3 = e^{-x}$ are linearly independent.

Here we must show that the equation (*) $A + Be^x + Ce^{-x} = 0$ for all x has ONLY got the trivial solution where A, B, C are all zero.

1st method. take (*) & its derivatives $\frac{d}{dx}(*)$, $\frac{d^2}{dx^2}(*)$ to obtain

$$\begin{array}{l} ① \\ ② \\ ③ \end{array}\left\{\begin{array}{l} A + Be^x + Ce^{-x} = 0 \\ Be^x - Ce^{-x} = 0 \\ Be^x + Ce^{-x} = 0 \end{array}\right\} \text{ for all } x \text{ ; these equations} \Rightarrow \begin{array}{l} ①-③ \\ ②+③ \\ ③-② \end{array}\left\{\begin{array}{l} A = 0 \\ 2Be^x = 0 \\ 2Ce^{-x} = 0 \end{array}\right. \text{ for all } x$$

note I only used $\Rightarrow$ & did not need $\Leftrightarrow$

$\Rightarrow A = B = C = 0$ (e.g. use $x = 0$). So any solution of (*) MUST be the trivial one $A = B = C = 0$ & $1, e^x, e^{-x}$ are linearly independent.

2nd method since (*) is true for all x, let's see what happens for $x = 0$, $x = 1$, $x = -1$:

$$(*) \Rightarrow \left\{\begin{array}{ll} A + B + C = 0 & ④ \text{ (special case } x=0) \\ A + Be + \frac{C}{e} = 0 & ⑤ \text{ (sp. case } x=1) \\ A + \frac{B}{e} + Ce = 0 & ⑥ \text{ (sp. case } x=-1) \end{array}\right. \Leftrightarrow \left\{\begin{array}{ll} A + B + C = 0 & ④ \\ B(e-1) + C(\frac{1}{e}-1) = 0 & ⑤-④ \leftarrow \text{mult. by } \frac{1}{(e-1)} \\ B(\frac{1}{e}-1) + C(e-1) = 0 & ⑥-④ \end{array}\right.$$

(not $\Leftrightarrow$ in general!)

$$\Leftrightarrow \left\{\begin{array}{l} A + B + C = 0 \\ B - \frac{C}{e} = 0 \\ -\frac{B}{e} + C = 0 \end{array}\right. \cdot \tfrac{+1}{e} \Rightarrow \left\{\begin{array}{l} A + B + C = 0 \\ B - \frac{C}{e} = 0 \\ (1 - \frac{1}{e^2})C = 0 \text{ but } 1 - \frac{1}{e^2} \neq 0 \text{ so } C = 0 \end{array}\right. \Rightarrow \text{so } B = 0 \Rightarrow \text{so } A = 0$$

and we're done.

6. (8 points) Consider but DO NOT SOLVE the differential equation

$$x^2(x-4)^2(x^2-4x+5)y'' + (x-1)(x-4)y' + y = 0.$$

Repeat: DO NOT SOLVE THIS EQUATION!

a) Identify the singular points (both real and complex) and indicate which singular points are regular singular points.

$$P = \frac{(x-1)(x-4)}{x^2(x-4)^2(x^2-4x+5)} = \frac{(x-1)}{x^2(x-4)(x^2-4x+5)}$$

$$Q = \frac{1}{x^2(x-4)^2(x^2-4x+5)}$$

note that the roots of x^2-4x+5 are $2 \pm i$, both simple roots. If you like, you can factor $x^2-4x+5 = (x-\alpha)(x-\beta)$ with $\alpha = 2+i$, $\beta = 2-i$

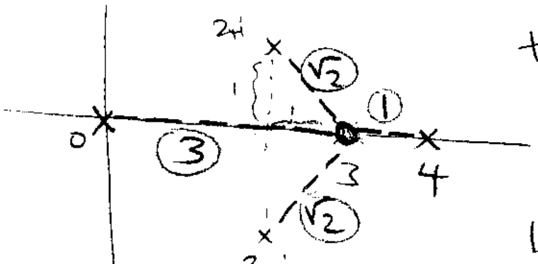
(1) the singular points are those where either P or Q has a problem, i.e. these are the roots of the denominators of either one. These are $\boxed{x=0,\ x=4,\ x=2+i,\ x=2-i}$

(2) the regular singular points are those singular points $x=a$ such that the denominator of P contains at most one factor $(x-a)$ & the denominator of Q contains at most the square factor $(x-a)^2$. The only hiccup here is that P has x^2, not x in the denominator (too high a power of x). So

$\boxed{\text{the regular singular points are } x=4,\ x=2+i,\ x=2-i}$

b) WITHOUT SOLVING THE DIFFERENTIAL EQUATION, find the minimum guaranteed radius of convergence of a series solution of the form $y = \sum_{n=0}^{\infty} c_n(x-3)^n$. Justify your reasoning.

Look at the center $x=3$ & the singular points are $x=0$, $x=4$, $x=2\pm i$ in the complex plane.



the distances from 3 to each singular point are circled. Here $1 < \sqrt{2} < 3$ & so

1 = the distance from the center $x=3$ to the nearest singular point (i.e. $x=4$)

By the theorem, we are guaranteed a minimum guaranteed radius of convergence of $\underline{\underline{1}}$.

(So our solution $\sum c_n(x-3)^n$ is guaranteed to converge for $|x-3| < 1$.)