STAT 230. Fall 2016-T2

Multiple Choice Questions Five (5) points are assigned to each correct answer. No penalty for wrong answers. You may select more than one answer. If you decide to do, you must *distribute* the points among the selected answers. Indicate in the empty box beside the answer the (integer) number of points you wish to allocate to that answer.

- Approximate using an appropriate distribution the following

$$\sum_{i=39}^{44} \binom{400}{i} \left(\frac{1}{10}\right)^{i} \left(\frac{9}{10}\right)^{400-i}$$

  A [ ] 0.307
  B [ ] 0.367
  C [X] 0.372
  D [ ] 0.345

- Let $X \sim \mathcal{N}(\mu = 10, \sigma^2 = 25)$ and $Y \sim \mathcal{N}(\mu = 20, \sigma^2 = 25)$ be two independent random variables. Compute $P(-12.5 < 12X - 5Y < 85)$.

  A [ ] 0.435
  B [ ] 0.398
  C [X] 0.533
  D [ ] 0.412

- Let X and Y be two random variables with the following joint p.m.f.

$p(x,y)$		$y=0$	$y=1$
x	0	1/2	1/4
	1	1/4	0

  Find $\rho(X,Y) \equiv Cor(X,Y)$.

  A [ ] 0
  B [ ] +1/4
  C [ ] -1/4
  D [X] -1/3

- Which of the following statements is true?

  A [ ] Since a standard normal density is symmetric about zero, $\Phi(-1) = \Phi(1)$.
  B [ ] If $Z \sim \mathcal{N}(0,1)$, then the random variable $Y = Z^2$ is $\mathsf{Exp}(\lambda = 1)$.
  C [ ] The chi-squared density function is symmetric about the mean.
  D [X] The points of inflection of the standard normal density curve $f_Z(z)$ are located at $z = \pm 1$.

- Let $X \sim \mathsf{Gamma}(\alpha = 2, \beta = 2)$. Which of the following is the p.d.f. of $Y = \log(X)$? [Notation: $\log(x) \equiv \ln(x)$, $\exp(x) \equiv e^x$]

  A [X] $f_Y(y) = \exp(2y)\exp(-\exp(y)/2)/4,\ y \in \mathbb{R}$

  B [ ] $f_Y(y) = \exp(2|y|)\exp(-\exp(y)/2)/4,\ y \in \mathbb{R}$

  C [ ] $f_Y(y) = \exp(-\exp(-y)/2)/4,\ y > 0$ and $f_Y(y) = 0, y \leq 0$

  D [ ] $f_Y(y) = \log(|y|) - y/2 - \log(4),\ y \in \mathbb{R}$

- Let $X \sim N(\mu = 3.5, \sigma^2 = 1)$ and $Y \sim \mathsf{Gamma}(\alpha = 2, \beta = 2)$ be independent random variables. What is the probability that the maximum of X and Y is less than 4?

  A [X] 0.41

  B [ ] 0.34

  C [ ] 0.47

  D [ ] 0.37

- Which of the following statements is false?

  A [ ] If X and Y are two different continuous random variables, $P(X = Y) = 0$.

  B [ ] If X and Y are continuous i.i.d. random variables, $P(X > Y) = P(X < Y)$.

  C [ ] For any pair of random variables X and Y, $|X - Y| = \max\{X, Y\} - \min\{X, Y\}$.

  D [X] The joint p.d.f. of two continuous random variables X and Y can be computed with the knowledge of the conditional density function $f_{Y|X}$ and of the marginal density function f_Y.

- Let $X \sim \mathsf{Geo}(p = 0.3)$ and $Y \sim \mathsf{Geo}(p = 0.2)$ be independent geometric random variables. The probability that $P(X = Y)$ is

  A [ ] 0

  B [X] 0.136

  C [ ] 0.111

  D [ ] 0.078

- The stress in certain bridge connections follows an exponential distribution with mean value 6.11 MPa. A stress test is carried out. Given that the stress is at least 6.56 MPa, what is the expected stress (in MPa)?

  A [ ] 9.46
  B [ ] 6.56
  C [ ] 6.11
  D [X] 12.67

- After having successfully passed the STAT230 exam, Rabeai decides to sell his statistics book. He will sell it to the first person who will offer at least 32 dollars. Assume that the offers are independent chi-squared random variables with mean 23 dollars. What is the expected number of offers he will have?

  A [ ] 9
  B [X] 10
  C [ ] 11
  D [ ] 12

- Two regular and fair dice are rolled. Let X and Y be the numbers that come up. Compute $Cov(2X - Y, X + 3Y - 2)$.

  A [ ] -5.84
  B [ ] -8.76
  C [X] -2.92
  D [ ] -12.25

- Let X be a random variable with the following c.d.f.

$$F(x) = \begin{cases} 0 & x < -2 \\ \frac{x+2}{5} & -2 \le x < -1 \\ \frac{x+3}{5} & -1 \le x < 1 \\ \frac{4}{5}x^2 & 1 \le x < \sqrt{5}/2 \\ 1 & x \ge \sqrt{5}/2 \end{cases}$$

  Find $E(X)$.

  A [ ] -0.245
  B [X] -0.288
  C [ ] -0.112
  D [ ] -0.088

- Let $X \sim \mathcal{N}(\mu = 50, \sigma^2 = 25)$. Find the probability that X exceeds 52 by at least one standard deviation.

  A [ ] 0.0937

  B [ ] 0.0832

  C [X] 0.0808

  D [ ] 0.0793

Problem (4 questions. 20 points)

Let x be a point chosen from a uniform distribution in the interval $[-1, 2]$. Let us then choose a point y uniformly from the interval $[0, x^2]$. Namely, $X \sim U[-1, 2]$ and $Y|X = x \sim U[0, x^2]$.

a) *Show* whether or not (x, y) is a random sample from a uniform distribution in an appropriate region of $\mathbb{R}^2$ and find this region. [That is, determine the joint density of (X, Y).]

The joint pdf is determined by $f_{Y|X}(y|x)$ and $f_X(x)$:

$$f(x, y) = f_{Y|X}(y|x) f_X(x).$$

Here $f_X(x) = \frac{1}{3}\mathcal{I}(-1 \le x \le 2)$ and $f_{Y|X}(y|x) = \frac{1}{x^2}\mathcal{I}(0 < y \le x^2)$ (where $\mathcal{I}(A) = 1$ if A is true, and zero if A is false). Thus,

$$f(x, y) = \begin{cases} \frac{1}{3x^2} & -1 \le x \le 2, \quad 0 < y \le x^2 \\ 0 & \text{elsewhere} \end{cases}$$

The joint distribution is not uniform.

b) Compute the marginal density of Y.

The support of the distribution is the region of $\mathbb{R}^2$ bounded by $y = x^2$, the x axis, and the lines $x = -1$, $x = 2$. To determine the marginal of y, we need to consider two regions: $0 < y < 1$ and $1 \le y \le 4$. When $0 < y < 1$, x varies in $[-1, -\sqrt{y}] \cup [\sqrt{y}, 2]$. When $1 \le y \le 4$, x is in the interval $[\sqrt{y}, 2]$. Thus

$$f_Y(y) = \begin{cases} \int_{\sqrt{y}}^{2} \frac{1}{3x^2} dx + \int_{-1}^{-\sqrt{y}} \frac{1}{3x^2} dx = \frac{2}{3\sqrt{y}} - \frac{1}{2} & 0 < y < 1 \\ \int_{\sqrt{y}}^{2} \frac{1}{3x^2} dx = \frac{1}{3\sqrt{y}} - \frac{1}{6} & 1 \le y \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

c) Given that $Y = 1/2$, compute the probability that X is positive.

The conditional density of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)};$$

when the given y is in $0 < y < 1$

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{x^2}\left(\frac{2\sqrt{y}}{4-3\sqrt{y}}\right) & x \in [-1, -\sqrt{y}] \cup [\sqrt{y}, 2] \\ 0 & \text{elsewhere;} \end{cases}$$

when the given y is in $1 \le y < 4$,

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{x^2}\left(\frac{2\sqrt{y}}{2-\sqrt{y}}\right) & x \in [\sqrt{y}, 2] \\ 0 & x \notin [\sqrt{y}, 2]; \end{cases}$$

and $f_{X|Y}(x|y) = 0$ for other given values of y. It follows

$$\begin{aligned} P(X > 0|Y = 1/2) &= \frac{2\sqrt{1/2}}{4 - 3\sqrt{1/2}} \int_{\sqrt{1/2}}^{2} dx \, \frac{1}{x^2} \\ &= \frac{2\sqrt{1/2}}{4 - 3\sqrt{1/2}} \left(\sqrt{2} - \frac{1}{2}\right) = \frac{2\sqrt{2} - 1}{4\sqrt{2} - 3}. \end{aligned}$$

d) Knowing that Y is equal to 3, compute the expected value of X.

Using the conditional pdf obtained above,

$$\begin{aligned} E(X|Y=3) &= \frac{2\sqrt{3}}{2-\sqrt{3}} \int_{\sqrt{3}}^{2} dx\, x \cdot \frac{1}{x^2} \\ &= \frac{2\sqrt{3}}{2-\sqrt{3}} \log\left(\frac{2}{\sqrt{3}}\right). \end{aligned}$$

Problem (3 questions. 15 points)

Buses of the Poisson Company arrive at a bus stop according to a Poisson process with a rate of 1 bus every ten minutes. Tarek arrives at the bus stop at a random time.

a) What is the probability that he has to wait more than 10 minutes to take the next Poisson Company bus?

The time between two consecutive events in a Poisson process is an exponential random variable, with parameter λ equal to the rate of the process. Thus $\lambda = 1/10$ (using here and in the following minutes as units of time). Let T be the time Tarek will wait until the next Poisson bus. By the memoryless property of the exponential, T is exponential as well: $T \sim \mathsf{Exp}(\lambda = 1/10)$. Hence

$$P(T > 10) = e^{-10/10} = e^{-1}.$$

b) A second bus company, Punctual, serves the same route. A Punctual bus arrives at the bus stop exactly every 10 minutes. What is the probability that exactly two Poisson buses arrive between two consecutive buses of Punctual?

The time interval between two consecutive Punctual buses is fixed and equal to $\Delta t = 10$. The number $N(\Delta t)$ of events in a time interval Δt is a Poisson r.v. with mean $\lambda \Delta t$.

$$P(N(\Delta t) = 2) = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^2}{2!} = \frac{e^{-1}}{2}.$$

c) At his arrival, Tarek decides to get on the bus that will come first. What is the probability that he will board a Poisson bus, assuming bus arrivals are independent?

Tarek arrives at a random instant. The time Y to the next Punctual bus is thus a $U[0,10]$ random variable. He will end up taking the Poisson bus, if the Poisson bus arrives before the Punctual bus: namely, $T < Y$, with T as in a). T and Y are independent random variables, thus the joint pdf is the product of the two marginal pdf's.

$$\begin{aligned} P(T < Y) &= \int dt\, dy f_T(t) f_Y(y) = \int_0^{10} \frac{1}{10} dy \int_0^y dt\, \lambda e^{-\lambda t} \\ &= 1 - \frac{1}{10\lambda} + \frac{1}{10\lambda} e^{-\lambda 10} \\ &= e^{-1}. \end{aligned}$$