STAT 230. Test 1-Summer 2015

Name: Lecturer:

Two points are assigned to each correct answer. No penalty for wrong answers.

T **F** The mean is a better measure of center/location than the median, whenever the distribution of the data is highly skewed

T F If A and B are independent events, then their complements $\bar{A}$ and $\bar{B}$ are independent

T **F** For any three events A, B, C such that $P(C) > 0$, $P(A \cap B|C) = P(A|C) + P(B|C) + P(A \cup B|C)$

T **F** If $P(A) = P(B) = 0.8$, then $P(A \cap B) < 0.6$

T F Let X be a random variable with mean $\mu = 15$ and standard deviation $\sigma = 10$. According to Chebyshev's inequality, $P(-10 < X < 40) \geq 0.84$

T F If the sample variance is zero, all the observations are equal

T F The more the data are concentrated, the smaller the fourth spread/interquartile range

T **F** Knowing the first quartile (lower fourth) and third quartile (upper fourth) is not sufficient to determine whether a sample point is an outlier

Multiple Choice Questions. Four points are assigned to each correct answer. No penalty for wrong answers.

- Let X be a random variable with the following p.d.f.

$$f(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ \frac{1}{8} & 3 \le x \le 9 \\ 0 & \text{elsewhere.} \end{cases}$$

  The third quartile (75% percentile) of the distribution is

  A 3

  B 6.5

  C 7

  D 7.5

- Let X denote the vibratory stress (psi) on a wind turbine blade at a particular wind speed in a wind tunnel. Assume that the pdf of X is as follows

$$f(x) = \begin{cases} \frac{x}{\theta^2}\, e^{-\frac{x^2}{2\theta^2}} & x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

  Determine the 80th percentile of the vibratory test, when $\theta = 100$

  A 135

  B 179

  C 203

  D 233

- An urn contains 3 white balls and 7 red balls. Balls are drawn from the urn one by one without replacement. What is the probability that the first white ball is seen after the 6th draw?

  A $\frac{3}{10}$

  B $\frac{1}{30}$

  C $\frac{1}{40}$

  D $\frac{3}{4}$

- A random variable has the following c.d.f.

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.15 & 1 \le x < 2 \\ 0.25 & 2 \le x < 5 \\ 0.60 & 5 \le x < 8 \\ 1 & x \ge 8. \end{cases}$$

  What is its mean?

  A 4.5

  B 4.8

  C 5.3

  D 5.6

- In Lebanon, 60% of drivers wear a seat belt. What is the probability that in a random group of 20 drivers, there are between 10 and 15 (these numbers included) drivers who wear it?

  A 0.245

  B 0.821

  C 0.704

  D 0.128

- Find the probability that if we roll a die 6 times, we find exactly two numbers repeated twice (*e.g.* 121234, 335422)

  A 0.347

  B 0.058

  C 0.231

  D 0.694

- If people can be born with the same probability any day of the week, what is the probability that in a random group of seven people two were born on Monday and two on Sunday?

  A $\frac{7!5}{2!2!3!7^7}$

  B $\frac{5^3}{7^7}$

  C $\frac{7!5^3}{7^7}$

  D $\frac{7!5^3}{24 \cdot 7^7}$

- Five identical (indistinguishable) balls are to be randomly distributed into 4 distinct boxes. What is the probability that exactly one box remains empty?

  A 1/7

  B 2/7

  C 3/7

  D 5/7

- The height of a person selected at random from a large population is normally distributed with a mean of 170 cm and a standard deviation of 8 cm. What value of the height represents the shortest 30% of the population?

  A 160

  B 166

  C 170

  D 174

- A box contains 10 radio tubes, 4 bad and 6 good. Two tubes were drawn at random without replacement. Knowing that at least one of them is good, what is the probability that both tubes are good?

  A 5/13

  B 5/9

  C 1/2

  D 1/4

- Let X and Y be two independent and identically distributed exponential random variables, with mean 2. The variance of $Z = \min\{X, Y\}$ is

  A 1/4

  B 1/2

  C 1

  D 4

Problem. [17 points (5,5,2,5)]
Let X and Y be two random variables whose joint p.d.f is

$$f(x,y) = \begin{cases} \frac{1}{3}x & y \leq x \leq y+1, 0 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the marginal density of X

(b) Compute $Cov(X,Y)$

(c) Determine if X and Y are independent

(d) Find the expected value $E(Y|X=1/2)$

$$f_X(x) = \begin{cases} \frac{x}{3}\int_0^x dy = \frac{x^2}{3} & 0 < x < 1 \\ \frac{x}{3}\int_{x-1}^x dy = \frac{x}{3} & 1 \leq x < 2 \\ \frac{x}{3}\int_{x-1}^2 dy = \frac{x}{3}(3-x) & 2 \leq x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} E(XY) &= \int_0^2 dy\, y \int_y^{y+1} dx\, \frac{1}{3}x^2 = \frac{22}{9} \\ E(X) &= \int_0^2 dy \int_y^{y+1} dx\, \frac{1}{3}x^2 = \frac{16}{9} \\ E(Y) &= \int_0^2 dy\, y \int_y^{y+1} dx\, \frac{1}{3}x = \frac{11}{9} \\ Cov(X,Y) &= E(XY) - E(X)E(Y) = \frac{22}{9} - \frac{176}{81} = \frac{22}{81} \neq 0 \end{aligned}$$

thus X and Y are not independent. Since $f(x,y)$ is a function of x only,

$$f_{(Y|X=1/2)}(y) = \frac{f(x=1/2,y)}{f_x(1/2)} = const$$

thus $Y|X=1/2 \sim U[0,1/2]$ $(y \leq x = 1/2)$ and the expected value is $1/4$.

Problem[13 points]
A tour guide organizes hiking trips to the top of Mount Lebanon. Sometimes, because of bad weather, the guide decides to interrupt the excursion before reaching the mountain top.
When it rains but there are no strong winds, the guide interrupts the excursion 20% of the times.
In presence of strong winds, but no rain, the guide interrupts the excursion 30% of the times.
If it rains and there are strong winds, the guide interrupts the excursion 80% of the times.
Finally, if the weather is fair (no rain and no strong winds) the guide will always lead the hikers to the mountain top.
Assume the events "it rains along the trail path" and "there are strong winds along the path" to be independent and have probability 0.3 and 0.2 respectively.
You are told that on March 3rd, the guide led a group of tourists on the hiking trip but the group never made it to the mountain top. What is the probability that it rained along the path during that excursion?
Solution
$A =$"the mountain top is not reached"
$B =$"it rains along the path"
$C =$"there are strong winds"
We know that

$$\begin{aligned} P(B) &= 0.3 \\ P(C) &= 0.2 \\ P(B \cap C) &= P(B)p(C) = 0.06 \\ P(A|B \cap \bar{C}) &= .20 \\ P(A|C \cap \bar{B}) &= .30 \\ P(A|\bar{B} \cap \bar{C}) &= 0 \\ P(A|B \cap C) &= 0.8 \end{aligned}$$

Hence
$$P(B|A) = P(B \cap C|A) + P(B \cap \bar{C}|A)$$
By Bayes' theorem and by the independence of the events B, C (thus of $\bar{B}$, C; $\bar{B}$, $\bar{C}$; B, $\bar{C}$)

$$P(B|A) = \frac{P(A|B \cap C)P(B)p(C)}{P(A)} + \frac{P(A|B \cap \bar{C})P(B)p(\bar{C})}{P(A)}$$

with

$$\begin{aligned} P(A) &= P(A|B \cap C)P(B)p(C) + P(A|\bar{B} \cap C)P(\bar{B})p(C) \\ &\quad + P(A|\bar{C} \cap B)P(\bar{C})p(B) + P(A|\bar{B} \cap \bar{C})P(\bar{B})p(\bar{C}) \\ &= (8 \cdot 2 \cdot 3 + 3 \cdot 7 \cdot 2 + 2 \cdot 8 \cdot 3)/10^3 \end{aligned}$$

Thus

$$P(B|A) = 16/23$$

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Problem[10 points (2,4,4)]

In a multinational consultancy firm, each consultant is granted a mobile phone. The IT center is responsible for updating the software of those phones as soon as a phone with an expired software arrives at the IT help-desk. Suppose that phones arrive at the help-desk according to a Poisson process with a rate of 25 phones per working week (from Monday to Friday).

a) On average, how many phones would need a software update on a given day?

b) What is the probability that at least one phone needs an update on a given day?

c) What is the probability that at least ten phones arrive during the next two working days?

Solution

$\lambda = \frac{25}{5}\frac{1}{day} = \frac{5}{day}$

a)

$$E(N(t=1)) = \lambda \cdot 1 = 5$$

b)

$$P(N(t=1) \geq 1) = 1 - P(N(t=1)=0) = 1 - e^{-\lambda} = 1 - e^{-5}$$

c)

$$P(N(t=2) \geq 10) = 1 - P(N(t=2) \leq 9) = 1 - 0.458 = 0.542$$

or

$$\begin{aligned} P(\sum_{i=1}^{10} T_i \leq 2) &= P\left(X \leq 2, X \sim \mathsf{Gamma}(\alpha = 10, \beta = 1/5)\right) \\ &= P\left(X \leq 2 \cdot 5, X \sim \mathsf{Gamma}(\alpha = 10, \beta = 1)\right) = 0.542 \end{aligned}$$