American University of Beirut
Statistics 230
Test1, October 18^{th}, Fall 2012
Dr. Alhakim

NAME: KEY

You have 60 minutes to finish this test.

1. The cumulative distribution function of a random variable X is given as

$$\mathbf{F}(x) = \begin{cases} 0 & x < 2 \\ 1/4 & 2 \le x < 3 \\ 5/8 & 3 \le x < 5 \\ 5/16x - 11/16 & 5 \le x < 7 \\ 1 & x \ge 7 \end{cases}$$

Find $P[2 < X < 4]$ $= P[X < 4] - P[X \leq 2]$

$= F(4^-) - F(2) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$

2. A die is rolled four times. The probability that all numbers are the same is

total is 6^4, favorable cases: 6 (all 1, all 2, ..., all 6)

$\Rightarrow$ we get $6/6^4 = 1/6^3$.

OR: $P[\text{all } 1] + \cdots + P[\text{all } 6] = \underbrace{(1/6)^4 + \cdots + (1/6)^4}_{6} = (1/6)^3$.

3. Suppose A, B and C three mutually exclusive events such that $P(A) = 0.5$ and $P(B) = 0.4$, $P[C] = 0.6$, $P[B|A] = 0.6$, $P[C|A] = 0.8$, $P[B|C] = 0.4$ and $P[A|BC] = 0.75$. Find $P[A \cup B \cup C]$. $= P[A] + P[B] + P[C] - P[AB] - P[AC] - P[BC] + P[ABC]$

$P[AB] = P[A]P[B|A] = 0.3$

$P[AC] = P[A]P[C|A] = 0.4$

$P[BC] = P[C]P[B|C] = 0.24$

$P[ABC] = P[BC] \cdot P[A|BC]$

$= (0.24) \cdot (0.75) = 0.18$

$= 0.5 + 0.4 + 0.6$

$- 0.3 - 0.3 - 0.24$

$+ 0.18$

$= 0.74$

4. Find the number of distinct permutations of the word 'MARAYA' that start with the letter 'A'.

Put one A in the beginning & vary the rest :

in $\frac{5!}{2!\ 1!\ 1!\ 1!} = \frac{5!}{2} = 60.$

5. Find the probability that a four-letter word chosen randomly has exactly two consecutive letters that are identical (e.g. 'abbc', 'ccfd' etc.)

choose the repeated letter : 26 (say a)

choose 2 more without order $\binom{25}{2}$ (say b, c)

shuffle them (aabc, baac, bcaa).

in $3 \times 2!$ to get $26 \times \binom{25}{2} \times 3 \times 2! / 26^4$

6. The proportion of impurities in a batch of product of a chemical process has a probability density function given by

$$f(x) = \begin{cases} k(1-y)^5 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

A batch is not acceptable if the proportion of impurities exceeds 0.4. Calculate (numerically) the percentage of batches that are not acceptable. (Hint: you will need to first find the value of k.)

Find K: $1 = K\int_0^1 (1-y)^5 dy = K \left.\frac{(1-y)^6}{-6}\right|_0^1 = 0 + \frac{K}{6} = 1$

so $\boxed{K=6}$.

Now we want P[batch is not acceptable]

$= P[\text{proportion exceeds } 0.4] = \int_{0.4}^1 6(1-y)^5 dy$

$= -(1-y)^6 \Big|_{0.4}^1 = -0 + (0.6)^6 = (0.6)^6 = \cancel{0.736}$ 0.0466

7. A deck of cards is distributed equally between four players. Given that player 1 gets exactly 1 ace, what is the probability that player 2 gets all the other three aces?

$$P[\text{player 2 gets 3 aces} \mid \text{player 1 got exactly 1 ace}] = \frac{\binom{3}{3}\binom{36}{10}}{\binom{39}{13}}.$$

8. Select X uniformly from the set $\{1, 2, 3\}$. Then select Y uniformly from the set $\{1, \dots, X\}$. Find the expected value of Y.

$f_X(x) = \frac{1}{3}; x = 1,2,3,$ while $f_{Y|X}(y|x) = \frac{1}{x}; \; y = 1, \dots, x.$

$f(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x}; \; x = 1,2,3; \; y = 1, \dots, x.$

Get $f_Y(y)$ in order to get its expected value:

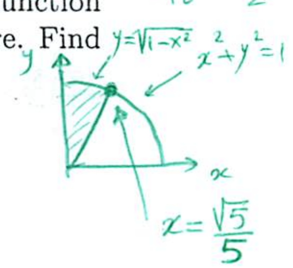
$f_Y(1) = f(1,1) + f(2,1) + f(3,1) = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$

$f_Y(2) = f(2,2) + f(3,2) = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

$f_Y(3) = f(3,3) = \frac{1}{9}$

So $EY = 1 \cdot \frac{11}{18} + 2 \cdot \frac{5}{18} + 3 \cdot 2/18 = 27/18 = \frac{3}{2}.$

9. Let X and Y be two continuous random variables with joint probability density function given by $f(x, y) = 3x$ when $0 < x < 1$, $0 < y < \sqrt{1 - x^2}$, and it is zero elsewhere. Find $P[Y > 2X]$.



$P[Y > 2X] = P[(X,Y) \in \text{shaded area}]$

$$= \int_0^{\sqrt{5}/5} \int_{2X}^{\sqrt{1-x^2}} 3x \, dy \, dx = \int_0^{\sqrt{5}/5} 3x\left(\sqrt{1-x^2} - 2x\right) dx.$$

10. A box with 12 chips contains either 0, 1, or 2 white chips with probabilities 0.1, 0.4 and 0.5 respectively. If among among 5 chips that are selected randomly exactly one is found to be white, what is the probability that this was the only white chip in the box?

Let A_0, A_1, A_2 be the events that there are 0, 1, OR 2 white chips respectively. Let B be the event that among 5 chips, exactly one is white. We want

$$P[A_1 \mid B] = \frac{P[A_1] \cdot P[B \mid A_1]}{P[A_0] \cdot P[B \mid A_0] + P[A_1]P[B \mid A_1] + P[A_2]P[B \mid A_2]}$$

$$= \frac{(0.4) \times \binom{1}{1}\binom{11}{4} / \binom{12}{5}}{(0.1) \cdot 0 + (0.4)\binom{1}{1}\binom{11}{4}/\binom{12}{5} + (0.5) \times \binom{2}{1}\binom{10}{4}/\binom{12}{5}}$$

$$= \frac{(0.4)\binom{11}{4}}{(0.4)\binom{11}{4} + (0.5) \times 2 \times \binom{10}{4}} = \frac{(0.4)(330)}{(0.4)(330) + 210}$$

$$= \frac{132}{342} = 0.386$$