American University of Beirut
Statistics 230
Final Exam, January 25^{th}, 2011
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NAME:

SECTION: Morning   Afternoon

PART I. Multiple choice questions.

1. The probability density function of X is given by $f(x) = 3x^2$; $0 < x < 1$. Find the cumulative probability function of $Y = 2X$
   (a) $3y^2/2$; $0 < y < 2$.
   (b) y^3; $0 < y < 2$.
   (c) $y^3/8$; $0 < y < 2$.
   (d) $3y^2/8$; $0 < y < 2$.
   (e) $y^3/2$; $0 < y < 2$.

   Answer is (c). Note that we want the c.d.f.

2. Which of the following statements is not true.
   (a) There exists a random variable whose mean is infinite.
   (b) There exists a random variable whose median is infinite.
   (c) There exists a random variable whose mean and median are identical.
   (d) There exists a random variable whose third quartile is smaller than its mean.

   Answer is (b). The median always exists while the mean can be infinite.

3. Let X_1 be uniform in the interval $[-a, a]$ and define $X_2 = X_1^2$. Then
   (a) X_1 and X_2 are independent and uncorrelated
   (b) X_1 and X_2 are dependent but uncorrelated
   (c) X_1 and X_2 are independent but correlated
   (d) X_1 and X_2 are dependent and correlated

   Answer is (b). Deterministically related, hence dependent, but their covariance is zero (do it!), hence uncorrelated.

4. Which of the following statements is false
   (a) The sum of two independent Poisson random variables is a Poisson random variable.
   (b) The sum of two independent gamma random variables is a gamma random variable.
   (c) The sum of two independent negative binomial random variables is a negative binomial random variable.
   (d) The difference of two independent normal random variables is a normal random variable.

   Answer is either (b) or (c): (b) is true iff both variables have the same parameter θ with possibly distinct α's. (c) is true iff both variables have the same parameter p with possibly different r's.

5. Given two independent exponential random variables X_1 and X_2. Which of the following statements is true?

(a) $Y_1 = min(X_1, X_2)$ is exponentially distributed.

(b) $Y_2 = max(X_1, X_2)$ is exponentially distributed.

(c) both of the above statements are correct.

(d) neither of the above statements is correct.

Answer is (a). Note that one has to find the distribution of Y_1 and Y_2 using the c.d.f. method and then answer this question.

6. Let X_1, X_2, X_3 be three independent $U[0, 1]$ random variables.

   The value of $Cov(X_1 + X_2, X_2 + X_3)$ is

   (a) 0 (b) 1/12 (c) 2/12 (d) 6/12

   Answer is (b). Splitting this covariance and using the independence of X_1, X_2, X_3 we get $cov(X_2, X_2)$ which is 1/12 because X_2 is $U[0, 1]$.

7. A machine disperses an amount of coke that is normally distributed with mean $\mu = 220$ milliliters and standard deviation 10 milliliters. Find the size of a can in milliliters so that no more than 1% of all cans are overfilled.

   (a) 220 (b) 222.3 (c) 232.3 (d) 243.3.

   Answer is (d). Letting s be the size of the can and X be the random amount of coke, we want s so that $P[X > s] = 0.01$. Standardizing we get $P[Z > \frac{s-220}{10}] = 0.01$. So by the normal table, $\frac{s-220}{10} = z_{0.01} = 2.326$ from which you get s.

8. For two random variables X and Y the following parameters are given. $\rho = 0.75$. $\sigma_X = 12$, $\sigma_Y = 6$, $\mu_X = 3$ and $\mu_Y = 1$. Write down a regression line (of best fit) to predict X using Y.

   (a) $\hat{x} = 3 + \frac{3}{2}(y - 1)$

   (b) $\hat{x} = 3 + \frac{3}{8}(y - 1)$

   (c) $\hat{y} = 1 + \frac{3}{2}(x - 3)$

   (d) $\hat{y} = 1 + \frac{3}{8}(x - 3)$

   Answer is (a). Note that we are regressing X on Y rather than Y on X as usual. So the slope $b = \rho \times \frac{\sigma_X}{\sigma_Y}$ (reversed).

PART II. Write your detailed solutions.

1. Find $P[X > 1/4]$ where X and Y are two random variables with the joint p.d.f.

$$f(x, y) = \begin{cases} 12xy & \text{for } 0 \le x \le 1, x^2 \le y \le \sqrt{x} \\ 0 & \text{otherwise} \end{cases}.$$

   Answer:

   $P[X > 1/4] = \int_{1/4}^{1} \int_{x^2}^{\sqrt{x}} 12xydxdy...$

2. Let $X_1, \ldots, X_{20}$ be independent chi square variables each with two degrees of freedom. Let $Y = X_1 + \ldots + X_{20}$. (a) By finding the distribution of Y find the exact probability $P[Y > 51.8]$

   Answer: Y is chi-square with 40 degrees of freedom. From the table we get the answer 0.1.

   (b) Use the central limit theorem to approximate the same quantity $P[Y > 51.8]$

   Answer: Since Y is a sum of iid random variables the central limit theorem applies with $\mu = 40$ and variance equal

80. (those of the chi square). Thus $P[Y > 51.8]$ is approximately $P[Z > (51.8 - 40)/\sqrt{80}]$ which is 0.0934 by the normal table.

3. A mathematics department sends one, two or three professors every year to an annual conference, which lasts for three days. The hotel at which the conference is held offers a reduced conference rate of \$150 per day per person if reservations are made 30 days or more in advance but it charges a flat cancelation fee of \$250. The regular rate of the hotel is \$300 per person per night. From past experience it is known that the probability of one, two or three professors attending the conference is $1/4, 1/2, 1/4$ respectively. Let X be the total cost in dollars if the department makes early reservations for *two* professors. Find $E[X]$.

   Answer: The department makes early reservations for *two* professors but at the end, one, two or three professors will attend.

   If one professor attends, the department has to pay a cancelation fee of 250 plus 450 for the professor who attends.

   If two professors attend, the department will pay 900.

   If three professors attend, they pay 900 at the reserved rate for two PLUS 900 for the third at regular price.

   The expected payment is then $(1/4) \times 700 + (1/2) \times 900 + (1/4) \times 1800$. This is 1075 USD.

4. Let X_1 and X_2 be two independent and exponentially distributed random variables with common parameter $\theta = 1$. Consider the transformation $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$.

   (a) Find the joint density of Y_1 and Y_2.

   (b) Are Y_1 and Y_2 independent?

   Answer (a):

   Since X_1 and X_2 are independent their joint pdf is $f(x_1, x_2) = e^{-x_1-x_2}$.

   The inverse transformation is $X_1 = Y_1Y_2$ and $X_2 = Y_1 - Y_1Y_2$. So the Jacobian can be easily calculated to be $|y_1| = y_1$.

   So the joint pdf of Y_1 and Y_2 is $g(y_1, y_2) = e^{-y_1} \times y_1$. This is true for $y_1 > 0$ and $0 < y_2 < 1$, this can be seen by noting that Y_1 is a sum of two positive unbounded variables and that Y_2 is a proportion of positive quantities such that the numerator is less than the denominator.

   (b) The marginals are

   $f_{Y_1}(y_1) = \int_0^1 y_1 e^{-y_1} dy_2 = y_1 e^{-y_1}$ for $y_1 > 0$ and $f_{Y_2}(y_2) = \int_0^\infty y_1 e^{-y_1} dy_1 = 1$ for $0 < y_2 < 1$.

   Hence Y_1 and Y_2 are independent.