American University of Beirut
STAT 230
Introduction to Probability and Random Variables
Summer 2009

Final Exam - solution

Exercise 1 Let X and Y be a couple of random variables with pdf

$$f(x,y) = cy \quad 0 < x < y < 1$$

a. find the value of the constant c

$\int_0^1 \int_0^y cydxdy = \int_0^1 cy^2dy = \frac{c}{3} = 1$ (cause f is a pdf), and hence $c = 3$

b. find the marginal pdf of X and Y. Are X and Y independent ?

$g(x) = \int_x^1 3ydy = \frac{3}{2}\left[y^2\right]_{y=x}^1 = \frac{3}{2}(1-x^2)\,, \qquad 0 < x < 1$

$h(y) = \int_0^y 3ydx = 3y^2\,, \qquad 0 < y < 1$

$f(x,y) \neq g(x) \times h(y)$, then X and Y are not independent.

c. find $P(X+Y > 1/4)$ (its advised to sketch the region of integration !)

$$P(X+Y > 1/4) = 1 - \int_0^{\frac{1}{8}} \int_x^{\frac{1}{4}-x} 3ydydx = 1 - \frac{3}{2}\int_0^1 \left(\frac{1}{16} - \frac{x}{2}\right) dx = 1 - \frac{3}{512} = \frac{509}{512}$$

d. find $E(X)$

$$E(X) = \int_0^1 xg(x)dx = \frac{3}{2}\int_0^1 (x - x^3)dx = \frac{3}{8}$$

Exercise 2 Let (X,Y) be a couple of random variables with joint pdf

$$f(x,y) = x + y\ ,\ 0 < x < 1\ ,\ 0 < y < 1$$

Let $U = X$ and $V = XY$. Find the joint pdf of the couple (U,V), then find the marginal pdf of V.

solving the system of equations yields $x = u$ and $y = \frac{v}{u}$

the Jacobian is $J(u,v) = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$

and hence

$$g(u,v) = 1 + \frac{v}{u^2} \qquad 0 \leq u \leq 1\ ,\ 0 \leq v \leq u$$

$h(v) = \int_v^1 (1 + \frac{v}{u^2})\, du = \left[u - \frac{v}{u}\right]_{u=v}^1 = 2 - 2v\,, \quad 0 < v < 1$

(*its advised to sketch the region of integration*)

Exercise 3 An electronic device runs until one of its three components fails. The lifetimes (in weeks), X_1, X_2, X_3, of these components are independent, and each has the pdf

$$f(x) = \frac{2x}{25}\, e^{-(x/5)^2} \ , \ 0 < x < \infty$$

Find the probability that the device stop running in the first three weeks.

$$p = 1 - (P(X_1 > 3))^3 = 1 - \left(\int_3^{\infty} \frac{2x}{25}\, e^{-(x/5)^2} dx \right)^3 = 1 - \left(\left[-e^{-(x/5)^2} \right]_3^{\infty} \right)^3 = 1 - e^{-27/25} = 0.66$$

Exercise 4 Let $X_1, \ldots, X_{18}$ be a random sample of size 18 with $\chi^2(1) = G(\frac{1}{2}, 2)$ distribution. Let $Y = X_1 + X_2 + \ldots + X_{18}$.

a. Find $P(9.39 < Y \leq 34.80)$

$Y \rightsquigarrow \chi^2(18)$ (the sum of independent χ^2 distribution is a χ^2 distribution), and hence

$P(9.39 < Y \leq 34.80) = 0.99 - 0.05 = 0.94$ (from the χ^2 table with $r = 18$)

b. Approximate $P(9.39 < Y \leq 34.80)$

$E(X_1) = 1$ and $Var(X_1) = 2$; by the central limit theorem $\frac{\frac{Y}{18} - 1}{\frac{\sqrt{2}}{\sqrt{18}}} = 3(\frac{Y}{18} - 1)$ has a standard normal distribution $\mathcal{N}(0, 1)$,

hence $P(9.39 < Y \leq 34.80) \simeq P(3(\frac{9.39}{18} - 1) < Z \leq 3(\frac{34.80}{18} - 1)) = P(-1.43 < Z \leq 2.8) = 0.9974 - (1 - 0.9236) = 0.921$

Exercise 5 Each item produced by a certain manufacturer is, independently, of acceptable quality with probability 0.95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Let X be the number of unacceptable items among the 150; X has a Binomial distribution $b(150; 0.05)$.

X can be approximated by a normal distribution $Y \rightsquigarrow \mathcal{N}(7.5; 7.125)$.

$P(X \leq 10) \simeq P(Y \leq 10) = P\left(\frac{Y - 7.5}{\sqrt{7.125}} \leq \frac{10 - 7.5}{\sqrt{7.125}} \right) = P(Z \leq 0.94) = 0.8264$ (from the standard normal table)