American University of Beirut

STAT 230

Introduction to Probability and Random Variables

Spring 2009

Final Exam - solution

Exercise 1 Let $X \rightsquigarrow \mathcal{U}(0,1)$. Find the pdf of $Y = \theta\sqrt{-\ln X}$ $(\theta > 0)$.
$g(x) = \theta\sqrt{-\ln x}$ is a one-to-one transformation; it maps $(0,1)$ into $(0,\infty)$.
$g^{-1}(y) = e^{(-y/\theta)^2}$, $(g^{-1})'(y) = -\frac{2y}{\theta^2}\, e^{-(y/\theta)^2}$,
and $h(y) = |(g^{-1})'(y)| \times f(g^{-1}(y)) = \frac{2y}{\theta^2}\, e^{-(y/\theta)^2}$, $0 < y < \infty$

Exercise 2 Let $f(x,y) = 2\, e^{-x-y}$, $0 \le x \le y < \infty$ be the joint pdf of X and Y. Are X and Y independent ?

$$g(x) = \int_x^\infty 2\, e^{-x-y}\, dy = 2e^{-2x}\,, \quad 0 < x < \infty$$

$$h(y) = \int_0^y 2\, e^{-x-y}\, dx = 2e^{-y}(1 - e^{-y})\,, \quad 0 < y < \infty$$

$g(x) \times h(y) \neq f(x,y)$, hence X and Y are not independent.

Exercise 3 Let X and Y be two independent random variables with exponential distribution with mean 1.

a. find the pdf of $U = X/Y$.

Consider $V = Y$, solving the system of equations yields $x = uv$ and $y = v$

the Jacobian is $J(u,v) = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

and hence

$$g(u,v) = v\, e^{-v(1+u)} \qquad 0 < u < \infty\ ,\ 0 < v < \infty$$

then $h(u) = \displaystyle\int_0^\infty v\, e^{-v(1+u)}\, dv = \frac{1}{(1+u)^2}\,,\ 0 < u < \infty$

b. find $P(X < Y)$.

$$P(X < Y) = P(U < 1) = \int_0^1 \frac{1}{(1+u)^2}\, du = \left[-\frac{1}{1+u}\right]_0^1 = \frac{1}{2}$$

Exercise 4 Let X_1, X_2, X_3 denote a random sample of size $n = 3$ from a distribution with the geometric pdf

$$P(X = k) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{k-1} \qquad k = 1, 2, 3, \ldots$$

a. find $P(X_1 = 1, X_2 = 3, X_3 = 1)$

$P(X_1 = 1, X_2 = 3, X_3 = 1) = P(X_1 = 1).P(X_2 = 3).P(X_3 = 1) = \frac{27}{1024}$

b. give the distribution of $Z = X_1 + X_2 + X_3$, then find $P(Z = 14)$

Z has a negative binomial distribution; $Z \rightsquigarrow NB(3; \frac{3}{4})$, and $P(Z = 14) = C_{13}^2 (\frac{1}{4})^3 (\frac{3}{4})^{11}$

c. let $Y = \min(X_1, X_2, X_3)$. Find $P(Y \leq n)$, then deduce the pdf of Y.

$P(Y \leq n) = 1 - P(Y > n) = 1 - P(\min(X_1, X_2, X_3) > n) = 1 - (P(X_1 > n))^3$

but $P(X_1 > n) = \displaystyle\sum_{k=n+1}^{\infty} \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^{k-1} = \left(\frac{1}{4}\right)^n$

(or $\{X_1 = n\}$ ={event the first n trials are all failures}, hence $P(X_1 > n) = (\frac{1}{4})^n$)

then $P(Y \leq n) = 1 - \left(\frac{1}{4}\right)^{3n}$

- $P(Y = n) = P(Y \leq n) - P(Y \leq n - 1) = \left(\frac{1}{4}\right)^{3n-3} - \left(\frac{1}{4}\right)^{3n} = 63. \left(\frac{1}{4}\right)^{3n}$, $n = 1, 2, 3, \ldots$

Exercise 5 The weight of a 6-pound box of soap has a normal distribution with mean 6.05 and variance 0.04. If nine boxes of soap are selected, find the probability that their total weight exceed 55.

Let $X_1, X_2, \ldots, X_9$ be the weight of the nine boxes respectively, and let $Y = X_1 + X_2 + \ldots + X_9$, hence $Y \rightsquigarrow \mathcal{N}(54.45; 0.36)$
$P(Y > 55) = P\left(\frac{Y-54.45}{0.6} > \frac{55-54.45}{0.6}\right) = P(Z > 0.92) = 1 - 0.8212 = 0.1788$

Exercise 6 Let $X_1, X_2, \ldots, X_{48}$ be a random sample of size 48 from the distribution with pdf $f(x) = \frac{1}{x^2}$, $1 < x < \infty$. Approximate the probability that at most 10 of these random variables have values greater than 4.

$$P(X_1 > 4) = \int_4^{\infty} \frac{1}{x^2}\, dx = 0.25.$$

Let Y be the number of variables that have values greater then 4 among the 48, then Y has a Binomial distribution $b(48; 0.25)$.

Y can be approximated by a normal distribution $X \rightsquigarrow \mathcal{N}(12; 9)$.

$P(Y \leq 10) \simeq P(X \leq 10) = P\left(\frac{X-12}{3} \leq \frac{10-12}{3}\right) = P(Z \leq -0.66) = P(Z > 0.66) = 1 - 0.7454 = 0.2546$