American University of Beirut STAT 230

Introduction to Probability and Random Variables Fall 2009-2010

Final Exam - solution

Exercise 1 Let X equal the weight of the soap in a 6-pound box. Assume the distribution of X is normally distributed with mean 6.05 and variance 0.0004

a) Find P(X < 6.0171)

<u>solution</u>: $P(X < 6.0171) = P(\frac{X-6.05}{\sqrt{0.0004}} < \frac{6.0171-6.05}{\sqrt{0.0004}}) = P(Z < -1.64) \simeq 0.05$ (from normal table)

b) If nine boxes of soap are selected at random form the production line, find the probability that at most two of the boxes weigh less than 6.0171 each

<u>solution</u>: Let Y be the number of boxes that weigh less than 6.0171, then $Y \rightsquigarrow b(9, 0.05)$ $P(Y \le 2) = 0.99$ (from binomial tables or use the binomial formula)

(Normal approximation to Binomial cannot be used here cause np < 5)

c) Find the probability that the mean weight of nine such boxes does not exceed 6.035 <u>solution</u>: $\overline{X} \rightsquigarrow \mathcal{N}(6.05, 0.0004/9)$, and

$$P(\overline{X} < 6.035) = P\left(\frac{\overline{X} - 6.05}{\sqrt{0.0004/9}} < \frac{6.035 - 6.05}{\sqrt{0.0004/9}}\right) = P(Z < -2.25) \simeq 0.01 \text{ (from normal table)}$$

Exercise 2 Let (X, Y) be a couple of random variables with pdf

$$f(x,y) = 4/3$$
, $0 < x < 1$, $x^3 < y < 1$

a) Find the marginal pdf of X and Y. Are X and Y independent ? *solution*:

$$-g(x) = \int_{x^3}^{1} \frac{4}{3} \, dx = \frac{4(1-x^3)}{3}, \ 0 < x < 1$$
$$-h(y) = \int_{0}^{y^{1/3}} \frac{4}{3} \, dx = \frac{4y^{1/3}}{3}, \ 0 < y < 1$$

- $f(x, y) \neq g(x) \times h(y)$, then X and Y are not independent.

b) Find
$$P(X > Y)$$

solution: $P(X > Y) = \int_0^1 \int_{x^3}^x 4/3 \, dy \, dx = \int_0^1 4(x - x^3)/3 \, dx = 1/3$
(it is advised to sketch the region of integration)

Exercise 3 Let (X, Y) be a couple of random variables with joint pdf:

$$f(x,y) = x e^{-x(y+1)}$$
, $x > 0$, $y > 0$

Find the pdf of Z = XY. solution:

- Consider T = X, solving the system of equations in z and t yields x = t and $y = \frac{z}{t}$ the Jacobian is $J(z,t) = \begin{vmatrix} 0 & 1 \\ \frac{1}{t} & -\frac{z}{t^2} \end{vmatrix} = -\frac{1}{t}$ and hence

$$g(z,t) = \frac{1}{t} \cdot t \cdot e^{t(z/t+1)} = e^{-(z+t)} \qquad 0 < z < \infty \ , \ 0 < t < \infty$$

$$h(z) = \int_0^\infty e^{-z} e^{-t} dt = e^{-z} \ , \quad 0 < z < \infty$$

Exercise 4 Let $X \rightsquigarrow \mathcal{N}(0,1)$ be a standard normal random variable with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
, $-\infty < x < \infty$

Consider $Y = X^2$

a) Find the pdf of Y

<u>solution</u>: Y has a $\chi^2(1)$ (done in class, but here is the solution again!)

 $y = x^2$ is one-to-one on each of the intervals $] - \infty, 0[$ and $]0, \infty[$; both intervals are mapped into $]0, \infty[$.

 $x = \sqrt{y} = g^{-1}(y)$, and $(g^{-1}(y))' = \frac{1}{2\sqrt{y}}$, and then $h(y) = 2 \cdot \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y/2} = \frac{1}{\sqrt{2\pi y}} \cdot e^{-y/2}$, $0 < y < \infty$ is the pdf of Y

b) Find $M_Y(t)$, the moment generating function of Y

<u>solution</u>: $Y \rightsquigarrow \chi^2(1)$, then $M_Y(t) = \frac{1}{\sqrt{1-2t}}$, t < 1/2 (done in class, but again here is the solution!)

$$M_Y(t) = E(e^{tY}) = E(e^{tX^2}) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-(1-2t)x^2/2} dx$$
$$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \sqrt{1-2t} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-(1-2t)x^2/2} dx$$

The later integral is equal to 1 (pdf of a normal distribution with mean 0 and standard deviation $\frac{1}{\sqrt{1-2t}}$), hence

$$M_Y(t) = \frac{1}{\sqrt{1-2t}}$$
, $t < 1/2$

Exercise 5 Three components are placed in series. The time in hours to failure of each has pdf

$$f(x) = \frac{x}{500^2} \cdot e^{-x/500}$$
, $0 < x < \infty$

Let $Y = \min(X_1, X_2, X_3)$. a) Find the cdf of Y, the pdf of Y<u>solution</u>: $H(y) = P(Y \le y) = 1 - P(Y > y) = 1 - \left(P(X_i > y)\right)^3 = 1 - (1 - F(y))^3$, but $F(y) = \int_0^y \frac{t}{500^2} \cdot e^{-t/500} dt = 1 - \frac{1}{500} \cdot y \cdot e^{-y/500} - e^{-y/500}$ for y > 0, and hence $H(y) = 1 - (\frac{1}{500} \cdot y \cdot e^{-y/500} + e^{-y/500})^3$, and $h(y) = 3(\frac{1}{500} \cdot y \cdot e^{-y/500} + e^{-y/500})^2 \cdot \frac{y}{500^2} \cdot e^{-y/500}$, $0 < y < \infty$ b) Find P(Y < 300)

solution:
$$P(Y \le 300) = 1 - P(X_i > 300)^3 = 1 - (\frac{300}{500}e^{-300/500} + e^{-300/500})^3 \simeq 0.32$$

Exercise 6 In a collection of 40 batteries, 20 are of type A and 20 are of type B. Type A batteries last for an amount of time that has mean 50 and standard deviation 15; type B batteries last for an amount of time that has mean 30 and standard deviation 6. Approximate the probability that the total life of all 40 batteries exceeds 1700.

solution:

Let X_1, \ldots, X_{20} be the lifetimes of the 20 batteries of type A, and

let Y_1, \ldots, Y_{20} be the lifetimes of the 20 batteries of type B

we want to find $P(X_1 + \ldots + X_{20} + Y_1 + \ldots + Y_{20} > 1700)$,

by the central limit theorem, $X_1 + \ldots + X_{20}$ is approximately normally distributed with mean $20 \times 50 = 1000$ and variance $20 \times 225 = 4500$, and

 $Y_1 + \ldots + Y_{20}$ is approximately normally distributed with mean $20 \times 16 = 720$ and variance $20 \times 15 = 225$, hence

 $T = X_1 + \ldots + X_{20} + Y_1 + \ldots + Y_{20}$ is approximately normally distributed with mean 1600 and variance 5220.

 $P(T > 1700) = P\left(\frac{T - 1600}{\sqrt{5220}} > \frac{1700 - 1600}{\sqrt{5220}}\right) = P(Z > 1.38) = 0.0838 \text{ (from normal table)}$