American University of Beirut

STAT 230

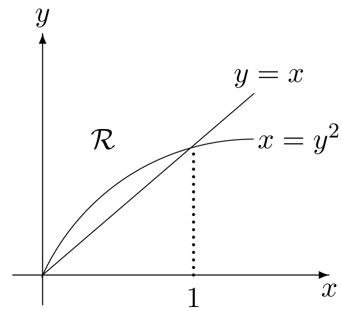
Introduction to Probability and Random Variables

Fall 2008-2009

Final Exam - solution

Exercise 1 Let $f(x, y) = 2$, $0 \le x \le y < 1$, be the joint pdf of X and Y. Find $P(X > Y^2)$.

$$P(X > Y^2) = \int_0^1 \int_{y^2}^{y} 2 \, dxdy = \int_0^1 2(y - y^2)dy = 1/3$$



Exercise 2 Let X_1, X_2, X_3 be independent random variables that represent lifetimes (in hours) of three key components of a device. Say their respective distributions are exponential with means $1000, 1500$, and 2000. Let $Y = \min(X_1, X_2, X_3)$. Find $P(Y > 1000)$.

$P(Y > 1000) = P(\min(X_1, X_2, X_3) > 1000) = P(X_1 > 1000) * P(X_2 > 1000) * P(X_3 > 1000)$
(cause X_1, X_2, X_3 are independent)
but if $X \mapsto f(x) = \frac{1}{\theta}e^{-x/\theta}$, then $P(X > y) = \int_y^\infty \frac{1}{\theta}e^{-x/\theta}dx = e^{-y/\theta}$;
hence $P(X_1 > 1000) = e^{-1}$, $P(X_2 > 1000) = e^{-2/3}$, and $P(X_3 > 1000) = e^{-0.5}$
and $P(Y > 1000) = e^{-1}e^{-2/3}e^{-0.5} = e^{-13/6}$

Exercise 3 A consumer buys n light bulbs, each of which has a lifetime that has a normal distribution with mean 800 hours, and a standard deviation of 100 hours. A light bulb is replaced by another as soon as it burns out. Assuming independence of the lifetimes, find the smallest n so that the succession of light bulbs produces light for at least 10000 hours with probability 0.8997.

Let $X_1, X_2, \ldots, X_n$ be the lifetimes of the n bulbs used in succession.
find n such that $P(X_1 + X_2 + \ldots + X_n > 10000) = 0.8997$; but $X_i \mapsto \mathcal{N}(800, 100^2)$, and $X_1 + X_2 + \ldots + X_n \mapsto \mathcal{N}(800n, 100^2 n)$
hence $P(X_1 + X_2 + \ldots + X_n > 10000) = P(\frac{X_1+X_2+\ldots+X_n-800n}{100\sqrt{n}} > \frac{100-8n}{\sqrt{n}}) = P(Z > \frac{100-8n}{\sqrt{n}}) = 0.8997$.
Form normal table, $\frac{100-8n}{\sqrt{n}} = -1.28$, and $n = 14$ by solving a quadratic equation.

Exercise 4 Let X and Y be a couple of random variables with pdf

$$f(x, y) = \frac{1}{x^2y^2} \qquad x \ge 1 \, , \; y \ge 1$$

a. find the joint pdf of $U = XY$ and $V = X/Y$

solving the system of equations yields $x = \sqrt{uv}$ and $y = \sqrt{\frac{u}{v}}$

the Jacobian is $J(u,v) = \begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \end{vmatrix} = -\frac{1}{2v}$

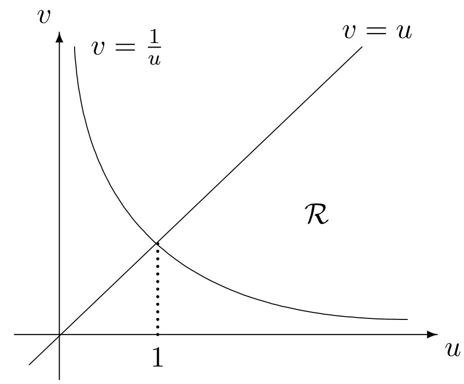
and hence

$$g(u,v) = \frac{1}{2u^2 v} \qquad uv \geq 1 \ , \ \frac{u}{v} \geq 1$$

b. find the marginal pdf of U and V

$$h(u) = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2 v} dv = \frac{1}{2u^2}\Big[\ln v\Big]_{\frac{1}{u}}^{u} = \frac{\ln u}{u^2} \qquad u \geq 1$$

$$k(v) = \begin{cases} \displaystyle\int_{1/v}^{+\infty} \frac{1}{2u^2 v}\, du = \frac{1}{2} & 0 \leq v \leq 1 \\ \displaystyle\int_{v}^{+\infty} \frac{1}{2u^2 v}\, du = \frac{1}{2v^2} & v \geq 1 \end{cases}$$



Exercise 5 Let X have a beta distribution with parameters α and β; the pdf of X is given by:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

$$E(X) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha}(1-x)^{\beta-1}dx = \frac{\alpha}{\alpha+\beta}\int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\, x^{\alpha}(1-x)^{\beta-1}dx = \frac{\alpha}{\alpha+\beta}$$

(the last integral is the pdf of a beta distribution with parameters $\alpha+1$ and β)

$$E(X^2) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha+1}(1-x)^{\beta-1}dx = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}\int_0^1 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)}\, x^{\alpha}(1-x)^{\beta-1}dx$$
$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

hence, $Var(X) = E(X^2) - (E(X))^2 = \dfrac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\dfrac{\alpha}{\alpha+\beta}\right)^2 = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)}$

Exercise 6 Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over the interval $(-1/2, 1/2)$. Find the probability that the resultant sum differs from the exact sum by more than 3.

Let $X_1, X_2, \ldots, X_{50}$ be the errors for the 50 numbers; $X_i \mapsto \mathcal{U}(-1/2, 1/2)$, $E(X_i) = 0$, and $Var(X_i) = 1/12$
$\frac{X_1+X_2+\ldots+X_{50}}{50} \sim \mathcal{N}(0, \frac{1}{600})$ by the central limit theorem.
$P(-3 < X_1 + X_2 + \ldots + X_{50} < 3) = P(-\frac{3\sqrt{600}}{50} < Z < \frac{3\sqrt{600}}{50}) = P(-1.47 < Z < 1.47) = 0.9292 - (1 - 0.9292) = 0.86$ (from the normal table)