

American University of Beirut
Math 204
Quiz I (Fall 2011)

Time 45 minutes.

Name: _____

ID#: _____

Instructor: N. Fuleihan

❖ Circle your problem solving section number below:

Section 13

8:00 Tu

Section 15

11:00 Tu

Section 14

9:30 Tu

Solution

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1) Given the matrix $A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -1 & -3 \\ 2 & 3 & -2 \end{pmatrix}$.

(6 pts) a) Calculate the determinant of A using the cofactors.

$$\begin{aligned} |A| &= 2C_{11} - 1C_{12} = 2(-1)^{1+1}M_{11} - 1(-1)^{1+2}M_{12} \\ &= 2 \begin{vmatrix} -1 & -3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix} = 2(2+9) + (-2+6) \\ &= 22+4 = 26 \end{aligned}$$

(6 pts) b) Consider the system $\begin{cases} 2x_1 - x_2 = 0 \\ x_1 - x_2 - 3x_3 = 2 \\ 2x_1 + 3x_2 - 2x_3 = 4 \end{cases}$ Use Cramer's Rule to find only x_1 .

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 2 & -1 & -3 \\ 4 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 0 \\ 1 & -1 & -3 \\ 2 & 3 & -2 \end{vmatrix}} = \frac{(-1)C_{12}}{26} = \frac{\begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix}}{26}$$

$$x_2 = \frac{-4+1^2}{26} = \frac{8}{26} = \frac{4}{13}$$

c) Let $B = \begin{pmatrix} 2 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 2 & 3 & 5 & 3 & 4 \\ 1 & 0 & -1 & 8 & -3 \\ 2 & 0 & 3 & 9 & -2 \end{pmatrix}$

(6 pts) Use the previous question a) to calculate the determinant of the matrix B

$$\begin{aligned} \begin{vmatrix} 2 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 2 & 3 & 5 & 3 & 4 \\ 1 & 0 & -1 & 8 & -3 \\ 2 & 0 & 3 & 9 & -2 \end{vmatrix} &= 4C_{24} = 4(-1)^{2+4}M_{24} = 4 \begin{vmatrix} 2 & 0 & -1 & 0 \\ 2 & 3 & 5 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 0 & 3 & -2 \end{vmatrix} \\ &= 4(3)C_{22} = 4(3)(-1)^{2+2}M_{22} = 12 \begin{vmatrix} 2 & -1 & 0 \\ 1 & -1 & -3 \\ 2 & 3 & -2 \end{vmatrix} \\ &= 12 \det(A) = 12(26) \end{aligned}$$

(6 pts)

- d) Deduce from the previous question a) the value of $\begin{vmatrix} 6 & -2 & -3 \\ 2 & -1 & -3 \\ 4 & 3 & -2 \end{vmatrix}$. Justify your answer.

$$\begin{vmatrix} 6 & -2 & -3 \\ 2 & -1 & -3 \\ 4 & 3 & -2 \end{vmatrix} = 2 \begin{vmatrix} 3 & -2 & -3 \\ 1 & -1 & -3 \\ 2 & 3 & -2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & 0 \\ 1 & -1 & -3 \\ 2 & 3 & -2 \end{vmatrix}$$

$R_1 - R_2$

$$= 2 |A| = 2(26)$$

(4 pts)

- e) Find $|5A|$

$$|5A| = 5^3 |A| = 5^3 (26)$$

- 2) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$. Find the matrix B such that $(B - 2C)^T = A^{-1}$.

(6 pts)

$$(B - 2C)^T = A^{-1}$$

$$(B - 2 \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix})^T = \frac{1}{-1-0} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$(B - \begin{pmatrix} 0 & 2 \\ -2 & 6 \end{pmatrix})^T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$B - \begin{pmatrix} 0 & 2 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$$

- 3) The sizes of the matrices A , B , C and D are 2×5 , 5×4 , 4×6 and 5×4 respectively and O is the zero matrix. Give the sizes of O and of each product if it is defined.

(10 pts)

$$\begin{aligned}
 ABC & \dots A \underset{2 \times 5}{\cancel{B}} C \underset{5 \times 4}{\cancel{C}} \underset{4 \times 6}{\cancel{D}} \text{ is } 2 \times 6 \text{ matrix} \\
 AOB & \dots A \underset{2 \times 5}{\cancel{O}} B \underset{5 \times 4}{\cancel{C}} \text{ is } 5 \times 5 \text{ matrix } AOB \text{ is } 2 \times 4 \\
 BCD & \dots B \underset{2 \times 5}{\cancel{C}} \underset{5 \times 4}{\cancel{D}} \text{ is not defined} \\
 BCO & \dots B \underset{2 \times 5}{\cancel{C}} O \underset{5 \times 4}{\cancel{D}} \text{ is } 6 \times n \text{ matrix } BCO \text{ is } 5 \times n \\
 & \quad \quad \quad \underset{5 \times 4}{\cancel{C}} \underset{4 \times 6}{\cancel{D}} \underset{6 \times n}{\cancel{O}}
 \end{aligned}$$

- 4) Let $B = (b_{ij})$ be a 2×5 for which $b_{ij} = \begin{cases} i+3j & \text{if } i < j \\ 5i & \text{if } i = j \\ 2-j+i^2 & \text{if } i > j \end{cases}$.

Determine b_{14} and the entries of the first row of B^T .

(6 pts)

$$b_{14} = 1+3(4) = 1+12 = 13$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{pmatrix} \quad B^T = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \\ b_{14} & b_{24} \\ b_{15} & b_{25} \end{pmatrix}$$

$$b_{11} = 5(1) = 5$$

$$b_{21} = 2-1+2^2 = 1+4 = 5$$

$$\boxed{b_{14} = 13} \quad \boxed{b_{11} = 5} \quad \boxed{b_{21} = 5}$$

- 5) Solve the matrix equation

$$(8 \text{ pts}) \quad 3 \begin{pmatrix} x+1 \\ y-2 \\ z+4 \end{pmatrix} - 2 \begin{pmatrix} 2x+1 \\ y+3 \\ z-2 \end{pmatrix} = \begin{pmatrix} x+5 \\ 3y+2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3x+3 \\ 3y-6 \\ 3z+12 \end{pmatrix} - \begin{pmatrix} 4x+2 \\ 2y+6 \\ 2z-4 \end{pmatrix} = \begin{pmatrix} x+5 \\ 3y+2 \\ 4 \end{pmatrix}$$

$$3x+3 - 4x - 2 = x+5 \rightarrow -x+1 = x+5 \rightarrow -2x = 4 \rightarrow \boxed{x = -2}$$

$$3y-6 - 2y-6 = 3y+2 \rightarrow y-12 = 3y+2 \rightarrow -y = 14 \rightarrow \boxed{y = -14}$$

$$3z+12 - 2z+4 = 4 \rightarrow z+16 = 4 \rightarrow \boxed{z = -12}$$

- 6) For which value(s) of m can you apply Cramer's rule to determine the unique solution

(8 pts) of the following system

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + 3x_2 + mx_3 = -2 \\ x_1 + mx_2 + 3x_3 = 2 \end{cases}$$

By Cramer's Rule the system has a unique solution if $D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & m \\ 1 & m & 3 \end{vmatrix} \neq 0$

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & m \\ 1 & m & 3 \end{vmatrix} = 1 \begin{vmatrix} 3 & m \\ m & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & m \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & m \end{vmatrix} \\ &= (9-m^2) - (6-m) - (2m-3) \\ &= (9-m^2) - 6 + m - 2m + 3 = (9-m^2) - 3 - m \\ &= (9-m^2) - (3+m) = (3-m)(3+m) - (3+m) \\ &= (3+m)[3-m-1] = (3+m)(2-m) \end{aligned}$$

$$D \neq 0 \rightarrow (3+m)(2-m) \neq 0 \rightarrow m \neq -3 \text{ and } m \neq 2$$

- 7) Determine the number of possible seven-digit telephone numbers if none of the first three digits can equal zero and:

- a. The last digit is 5

(12 pts)

$$\begin{array}{ccccccccc} 9 & 9 & 9 & 10 & 10 & 10 & 1 \\ 9 \times 9 \times 9 \times 10 \times 10 \times 10 \times 1 & = & 9^3 \times 10^3 \end{array}$$

- b. All digits must be even

$$\begin{array}{ccccccccc} 4 & 4 & 4 & 5 & 5 & 5 & 5 \\ 4 \times 4 \times 4 \times 5 \times 5 \times 5 \times 5 & = & 4^3 \times 5^4 \end{array}$$

- c. No digit can be repeated

$$\begin{array}{ccccccccc} 9 & 8 & 7 & 7 & 6 & 5 & 4 \\ 9 \times 8 \times 7 \times 7 \times 6 \times 5 \times 4 \end{array}$$

- 8) Consider the matrix $A = \begin{pmatrix} -1 & -1 & 2 \\ 0 & -1 & -2 \\ 2 & 0 & -3 \end{pmatrix}$ Find A^{-1} using the Gaussian elimination.

(22 pts)

Then use it to solve the system: $\begin{cases} -x_1 - x_2 + 2x_3 = 3 \\ x_2 + 2x_3 = 2 \\ 2x_1 - 3x_3 = 2 \end{cases}$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & -3 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & -3 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 2 & 0 \end{array} \right) \xrightarrow{R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & -2 & 1 & -2 & 0 & 1 \end{array} \right) \xrightarrow{R_3 + 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 5 & 2 & -2 & 1 \end{array} \right) \xrightarrow{\frac{1}{5}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{2}{5} & \frac{1}{5} \end{array} \right) \xrightarrow{R_1 + 4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{2}{5} & \frac{1}{5} \end{array} \right) \xrightarrow{R_2 - 2R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & -3/5 & 4/5 \\ 0 & 1 & 0 & -4/5 & -1/5 & -2/5 \\ 0 & 0 & 1 & 2/5 & -4/5 & 4/5 \end{array} \right) \quad A^{-1} = \begin{pmatrix} 3/5 & -3/5 & 4/5 \\ -4/5 & -1/5 & -2/5 \\ 2/5 & -4/5 & 4/5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -3 & 4 \\ -4 & -1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\begin{cases} -x_1 - x_2 + 2x_3 = 3 \\ x_2 + 2x_3 = 2 \\ 2x_1 - 3x_3 = 2 \end{cases} \rightarrow \begin{cases} -x_1 - x_2 + 2x_3 = 3 \\ -x_2 - 2x_3 = -2 \\ 2x_1 - 3x_3 = 2 \end{cases}$$

This system is $AX=B$ where $A = \begin{pmatrix} -1 & -1 & 2 \\ 0 & -1 & -2 \\ 2 & 0 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$$X = A^{-1}B$$

$$X = \frac{1}{5} \begin{pmatrix} 3 & -3 & 4 \\ -4 & -1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 23 \\ -14 \\ 12 \end{pmatrix}$$

$x_1 = \frac{23}{5}$
$x_2 = -\frac{14}{5}$
$x_3 = \frac{12}{5}$