AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department-FAS

MATH 251
FINAL EXAM
FALL 2010-2011

Closed Book, 2hours

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	15	
2	20	
3	15	
4	18	
5	20	
6	12	
TOTAL	100	

1. **(15 points)** Use of the standard formulae to solve the quadratic equation

$$(*) \quad x^2 - 10^3 x + 1 = 0 \quad \text{in } F(10, 5, -20, +20)$$

will cause a problem. Investigate this example, observe the difficulty, and propose a remedy to express the solutions x_1 and x_2 of the given equation in F, rounding to the closest. In this view:

(a) (5 pts) Compute the discriminant Δ of the quadratic equation $(*)$, in $F(10, 5, -20, +20)$:

(b) (5 pts) Compute the roots of $(*)$, in $F(10, 5, -20, +20)$:

(c) (5 pts) Suggested remedy to compute the solutions of $(*)$ in $F(10, 5, -20, +20)$:

2. **(20 points)** The reciprocal of the 3^{rd} root of a **negative integer** a (i.e. $r = \frac{1}{a^{1/3}}, a < 0$) can be computed by an iterative formula that does **not require division by the iterate**.

   (a) (5 pts)Establish this formula by applying Newton's method to some appropriate function $f(x)$, and draw the graph of $f(x)$.

   (b) (5 pts)Determine any necessary restrictions on the choice of the initial value of this iterative procedure.

(c) (5 pts)Prove that Newton's method is Quadratic.

(d) (5 pts)Prove that $r_{n+1} - r = (r_n - r)^2 F(r_n)$ where F is a quadratic polynomial in r_n, to be determined.

3. **(15 points)** Consider the following set of data

$D_n = \{(x_1, y_1), ..., (x_n, y_n) | y_i = f(x_i),\ \forall\ i = 1, ..., n\ \ ;\text{f : unknown function }\}$

where the x-nodes are equally spaced.

(a) (5 pts)

- Write the 1st order difference operator then the recurrence formula for the kth order difference operators

  $\Delta^1 y_i = \Delta y_i=$ ....; for i= ....

  for $k = 1$ to ....

  $\Delta^k y_i =$ ....; for i= ....

- Write the formula of Newton's interpolating polynomial $p_{12..n}(x)$ based on D_n using the difference operators.

  $p_{12..n}(x) =$

(b) (10 pts)Write a **MATLAB** function:

```
function[DF, pa]=DifferencesNewton(x,y,a)
```

which takes as input the 2 vectors x and y defined above and a real number a, and returns:

- a lower triangular matrix DF of size $n \times n$ whose entries are the differences of all orders based on the set D_n
- the real number $pa = p_{12\ldots n}(a)$ which represents the approximation of $f(a)$ using Newton's interpolating polynomial based on D_n.

```
function[DF, pa]=DifferencesNewton(x,y,a)
% Do NOT check any applicability condition.
n = length(x) ;
DF = zeros(n,n) ;

%  The entries of the first column of DF are the components of the
% y-vector
```

```
%  The remaining columns of DF are differences of all orders
```

```
% pa = Newton's Interpolating Polynomial evaluated at a
% HINT : if convenient, you may use the MATLAB built- in function
% computing  factorial of n :   factorial(n)










 end
```

4. **(18 points)** An integral of the form $I = \int_0^\infty f(x)\,dx$ can be transformed into an integral on a finite interval by making a change of variable. For instance, the substitution $x = -lny$ changes the integral $I = \int_0^\infty f(x)\,dx$ into $J = \int_0^1 g(y)\,dy$, where $g(0) = \lim_{y\to 0} g(y) = 0$.

   (a) (5 pts) Use this idea and write the integral J equivalent to the integral $I = \int_0^\infty [e^{-x}/(1+x^2)]\,dx$

   (b) (8 pts)Compute J by means of the Composite Trapezoidal rule, using 3 equally spaced partition points then find an upper bound for the error incurred in this approximation, knowing that $\max_{0\le y\le 1} |g"(y)| = M$.

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(c) (5 pts)Apply the $1st$ order Romberg scheme to improve the results obtained in part (b).

Express ALL the numerical results of this exercise in $F(10, 5, -15, +15)$**, rounding to the closest if needed.**

5. **(20 points)**

Let $D_n = \{(x_i, y_i)|i = 0, 1, ..., n = 2m, \text{ where } y_i = f(x_i))\}$ be a given set of data, where the x-coordinates are equally spaced, and where n is an even integer.

(a) (5 pts)Based on the set D_n, prove that the 1st Romberg operator applied to the composite Trapezoid rule is equivalent to Simpson's rule, that is prove that: $R^1(h) = S(h)$.

(b) (5 pts)Derive the first 2 Romberg approximation formulae: $S^1(h)$ and $S^2(h)$, **applied to the Composite Simpson's rule**, given that:

$$I = S(h) + s_1 h^4 + s_2 h^6 + ... + s_j h^{2j+2} + ...$$

(c) (10 pts)The next question deals with the following set of values for a function $f(x)$, arranged in a table as follows:

i	x_i	$f(x_i)$
0	0.000	1.0000
1	0.125	1.1108
2	0.250	1.1979
3	0.375	1.2663
4	0.500	1.3196
5	0.625	1.3600
6	0.750	1.3895
7	0.875	1.4093
8	1.000	1.4207

In order to approximate $\int_0^1 f(x)\,dx$, based on Simpson's rule and the formulae obtained in (a) and (b), fill in the empty slots of the following table adequately. <u>Express all your results in $F(10, 5, -15, +15)$</u>.

h	$T(h)$	$S(h)$	$S^1(h)$	$S^2(h)$
$h_0 = 1$				
$h_0/2 = 0.5$				
$h_0/4 = 0.250$				
$\frac{h_0}{8} = 0.1250$				

6. **(12 points)** Consider the following Initial value Problem:

$$(IVP)\begin{cases} \frac{dy}{dt} = t^3 + y\,; \;\; t \in [1, 2] \\ y(1) = 3 \end{cases}$$

To solve (IVP) in $F(10, 4, -20, +20)$ (rounding to the closest):

(a) (5 pts) (IVP) is first solved on $[1, 1.5]$ using the **discrete scheme** of Euler's method: (RK1).

- Write first the formulae for this scheme:

$$(RK1)\begin{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{i+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- Use **2 steps ONLY** of this scheme to approximate $y(1.25)$ and $y(1.50)$.

i	t_i	y_i	k_1	y_{i+1}
0	1.00	.	.	.
1	.	.	.	.
2	.	.	.	.

(b) (7 pts)(IVP) is then solved on [1.5,2] using the **discrete scheme** of the 2nd order Runge Kutta method - Midpoint Rule: (RK2.M).

- Write then the formulae of this scheme:

$$(RK2.M)\begin{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{i+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- Use **2 steps ONLY** of this scheme to approximate $y(1.75)$ and $y(2.00)$.

i	t_i	y_i	k_1	k_2	y_{i+1}
0	1.50	.	.	.	.
1	.	.	.	.	.
2	.	.	.	.	.