AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH-CMPS 251
FINAL EXAMINATION
FALL 2006-2007
Closed Book, Two hours

(ONLY NON-PROGRAMMABLE AND NON-GRAPHIC CALCULATORS ARE ALLOWED)

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	10	
2	15	
3	30	
4	15	
TOTAL	70	

1. (10 points) Loss of significant figures may result in the computation of the following functions of the variable x for certain values of x. Specify these values then propose alternativefunctions that would remedy the loss of significant figures. (If necessary you may use Taylor's series).

   (a) $f(x) = \ln(x + \sqrt{x^2 + 1})$

   (b) $g(x) = \sqrt{x^4 + 4} - 2$

2. (15 points) Solve the following:

   (a) (4 points) Let $p_j = x^{2^j}$, $j \in \mathbb{N}$, $j \geq 0$. Prove using mathematical induction, the recurrence:

$$p_j = (p_{j-1})^2,\ j \geq 1.$$

   (b) (6 points) Conclude the minimum number of arithmetic operations to compute (using whwn necessary nested evaluation):

   - x^{32}

   - $x^{32} + x^{40} + x^{50}$

   - $6(x+3)^2 + 9(x+3)^5 - 5(x+3)^8$

(c) (5 points) Fill in the missing statements in the following MATLAB program that computes using nested multiplications, the polynomial:

$$p(x) = a(1) + a(2)x + a(3)x^2 + ... + a(n+1)x^n$$

```
function p=nested(a,x)
%Input arguments:
%     vector a representing the coefficients: a(1),...,a(n+1)
%     a scalar x
%Output argument:
%      p=p(x) the value of the  polynomial at x

m=__________________; %get the number of elements of a
p=__________________; %initialize p
for i=____:____:____
     p=_____________
end
```

Find the number of arithmetic operations needed to execute such algorithm.

3. (30 points) In this problem, we consider the following table for the function $f(x)$. All computations shall be carried out **with 8 significant figures**.

x_i	y_i
1.000	1.58760060
1.125	1.68889110
1.250	1.80095954
1.375	1.92555928
1.500	2.06463973
1.625	2.22037684
1.750	2.39520718
1.875	2.59186604
2.000	2.81343020

(a) (5 points) Write the polynomial of degree 3, $p(x)$, that would best approximate $f(1.4)$. Find $p(1.4)$.

(b) (6 points) Using the central difference formula to approximate $f'(1.5)$, followed by Richardson's extrapolation find the best approximation to $f'(1.5)$. For that purpose, give the formulae that provide $\phi_{c,h}$, $\phi_{c,h}^{(1)}$, $\phi_{c,h}^{(2)}$, then fill out the table that follows.

- $\phi_{c,h}$=

- $\phi_{c,h}^{(1)}$=

- $\phi_{c,h}^{(2)}$=

h	$\phi_{c,h}$	$\phi_{c,h}^{(1)}$	$\phi_{c,h}^{(2)}$
0.5			
0.25			
0.125			

(c) (20 points) Consider the approximation of the integral $I = \int_0^1 f(x)dx$. Let:

$$x_i = ih,\ i = 0, 1, ...n, \quad (1)$$

be a subdivision of the interval $[0, 1]$ into n equi-spaced sub-intervals of size h.

i. (4 points) Give the formula for the composite trapezoidal rule $T(h)$ used to approximate I. Give an expression of $I - T(h)$ en terms of powers of h.

ii. (8 points) Prove the following relation:

$$T(\frac{h}{2}) = \frac{T(h)}{2} + \frac{h}{2}\Sigma_{k=0}^{n-1} f(m_k),$$

where $m_k = (x_k + x_{k+1})/2$ is the middle point of the interval (x_k, x_{k+1}) in the partition (1).

(Answer Sheet)

iii. (8 points) Give the formulae for $T^{(j)}(h)$, $j = 1, 2, 3$,

- $T^{(1)}(h)$=

- $T^{(2)}(h)$=

- $T^{(3)}(h)$=

then fill up the empty spaces of the table:

h	$T(h)$	$T^{(1)}(h)$	$T^{(2)}(h)$	$T^{(3)}(h)$
1				
0.5				
0.25				
0.125				

4. (15 points) Consider a second order differential equation given by:

$$y'' = y - 1, \;\; y(0) = 1, \;\; y'(0) = 0.5$$

(a) (5 points) Write this differential equation in the form

$$\vec{y}\,'(t) = \vec{F}\,(\vec{y}), \;\; \vec{y}\,(0) = \vec{y}_0 \,.$$

Specify $\vec{F}$ and $\vec{y}_0$.

(b) (5 points) Complete the formulae for a second order Runge Kutta method of the form:

$\vec{k_1} \quad = \tau \, \vec{F}\,(\vec{Y_i})$

$\vec{k_2} \quad = \tau$......................................

$\vec{Y_{i+1}} \; = \vec{Y_i} \, +$......................................

(c) (5 points) Find approximations to $y(0.125)$ and $y'(0.125)$ using such Runge Kutta method