Discrete Structures

Topic 3 – Logic: Propositional Equivalences (Ch 1.3)*

CMPS 211 – American University of Beirut

* Extracted from *Discrete Mathematics and It's Applications* book slides

Tautologies, Contradictions, and Contingencies

- A tautology is a proposition which is always true
  - Example: $p \lor \neg p$
- A contradiction is a proposition which is always false
  - Example: $p \land \neg p$
- A contingency is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logical Equivalence

- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology

- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions

Logical Equivalence (cont.)

- Example:
  - Show that $\neg p \vee q$ is equivalent to $p \rightarrow q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Proving Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements

- To prove that $A \equiv B$, we produce a series of equivalences beginning with A and ending with B

$$
\begin{aligned}
A &\equiv A_1 \\
&\vdots \\
A_n &\equiv B
\end{aligned}
$$

De Morgan's Laws

Augustus De Morgan

1806-1871

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

This truth table shows that De Morgan's Second Law holds

p	q	$\neg p$	$\neg q$	$(p \lor q)$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p\,, \quad p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T, \quad p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p\,, \quad p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T\,, \quad p \wedge \neg p \equiv F$

Key Logical Equivalences (cont.)

- Commutative Laws: $p \vee q \equiv q \vee p \,, p \wedge q \equiv q \wedge p$

- Associative Laws: $$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$ $$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

- Distributive Laws: $$(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$$ $$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

- Absorption Laws: $$p \vee (p \wedge q) \equiv p$$ $$p \wedge (p \vee q) \equiv p$$

More Logical Equivalences

TABLE 7 **Logical Equivalences Involving Conditional Statements.**

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Proofs

- Example:
  - Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$
- Solution:

$$
\begin{array}{lcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
 & \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
 & \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
 & \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
 & \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
 & \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
 & & & \text{for disjunction} \\
 & \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Equivalence Proofs

- Exercise:
  - Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Propositional Satisfiability

- A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true

- When no such assignments exist, the compound proposition is unsatisfiable

- A compound proposition is unsatisfiable if and only if its negation is a tautology

Propositional Satisfiability (cont)

- Example:
  - Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

    - Satisfiable by assigning T to *p, q,* and *r*

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

    - Satisfiable by assigning T to *p* and *F* to *q*

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

    - Not satisfiable
    - Check each possible assignment of truth values to the propositional variables and none will make the proposition true

Solving Satisfiability Problems

- In complex problems, for example Sudoku, you have very complex expressions consisting of hundreds of variables
- A truth table can always be used to determine the satisfiability of a compound proposition
  - But this is too time consuming even for modern computers for large problems
- For instance, there are $2^{20} = 1{,}048{,}576$ rows in the truth table of a compound proposition with just 20 variables

Solving Satisfiability Problems

- For a proposition with 1000 variables,
- There are 2^{1000} rows in the truth table
- Checking each one at a time cannot be done by a computer in even trillion years!!!
- However, progress has been made developing methods for solving the satisfiability problem that terminate quickly on average, i.e., most of the time
  - For example, in the case of Sudoko puzzles

Any Questions?