Discrete Structures

Topic 1 – Logic: Propositional Logic (Ch 1.1)*

CMPS 211 – Fall 2017 – American University of Beirut

* Extracted from *Discrete Mathematics and It's Applications* book slides

Why logic?

- Logic is a set of principles that can be used to reason about (mathematical) statements
- For instance, let's say we want to formally express and reason about the following statement:
  - "For every positive integer *n*, the sum of the positive integers not exceeding *n* is $n(n+1)/2$"
- We can formally express the above statement using logic
- We can also prove the above statement or argue whether it is true or false using logic

Why logic? (cont.)

- Logic has numerous applications to Computer Science
  - Used in the design of computer circuits
  - Used in the construction of computer programs
  - Used to verify the correctness of programs
  - Used to ensure the security of a system
  - Used heavily in artificial intelligence

Propositional Logic

Propositions

- A proposition is a declarative statement that is either true or false

- Examples of propositions:
  a) The Moon is made of white cheese
  b) Toronto is the capital of Canada
  c) A week has more days than a month
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$

- Examples that are not proposition
  a) Sit down! - *command*
  b) What time is it? - *question*
  c) $1 + 2$ - *expressions with a non-true/false value*
  d) This statement is false *– a paradox*

Atomic propositions

- We use letters (p, q, r, s, …) to denote atomic propositions
  - Also called propositional variables
  - Similar to x, y, z, … for numerical variables
  - For example, let p be the proposition that the earth is round and q be the proposition that the moon is flat
  - These represent a single statement that cannot be "decomposed"

Compound propositions

- Compound propositions are built up from atomic propositions by the use of Boolean connectives. Also called propositional formulae.
- Propositional Logic is the logic of propositional formulae and their meaning
  - First developed by the Greek philosopher Aristotle more than 2300 years ago
  - George Boole introduced Boolean Algebra in 1854

Propositional Connectives

Propositional Operators/Connectives

- An operator or connective combines one or more operand expressions into a larger expression (*e.g.*, "+" in numeric expressions)
  - *Unary* operators take 1 operand (*e.g.*, −3)
  - *binary* operators take 2 operands (*e.g.*, 3 × 4)

- Propositional or Boolean operators operate on propositions (or their truth values) instead of numbers

- There are six main operators
  - Negation $\neg$
  - Conjunction $\wedge$
  - Disjunction $\vee$
  - The Exclusive Or $\oplus$
  - Implication $\rightarrow$
  - Biconditional $\leftrightarrow$

Connectives: *Negation*

- The negation of a proposition p is denoted by $\neg p$
- In an English statement, we express $\neg p$ as follows: "It's not the case that p"
  - $\neg$p is true if p is false and is false if p is true
- Example:
  - If p denotes "I am at home",
  - then $\neg$p denotes "It is not the case that I am at home" or more simply "I am not at home"
- The negation of a proposition p has this truth table

p	$\neg p$
T	F
F	T

Connectives: *Conjunction*

- The conjunction of propositions p and q is denoted by $p \wedge q$
- It is the proposition "p and q"
- $p \wedge q$ is true when both p and q are true and is false otherwise

- **Example:**
  - If p denotes "I am at home", and
  - q denotes "It is raining"
  - then $p \wedge q$ denotes "I am at home and it is raining"

Conjunction (cont.)

- The conjunction of propositions p and q has this truth table

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Connectives: *Disjunction*

- The disjunction of propositions p and q is denoted by $p \vee q$
- It is the proposition "p or q"
- $p \vee q$ is false when both p and q are false and is true otherwise

- **Example**:
  - If p denotes "I am at home", and
  - q denotes "It is raining"
  - then $p \vee q$ denotes "I am at home or it is raining"

Disjunction (cont.)

- The disjunction of propositions p and q has this truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Connectives: *The Exclusive Or*

- The exclusive or of p and q is denoted by $p \oplus q$
- We say p *XOR* q
- $p \oplus q$ is true when exactly either p or q is true and is false otherwise

- **Example:**
  - If p denotes "I am at home", and
  - q denotes "It is raining"
  - then $p \oplus q$ denotes :"either I am at home or it is raining" but not both

The Exclusive Or (cont.)

- Note that English "or" can be ambiguous regarding the "both" case!
  - "Pat is a singer or Pat is a writer" $\vee$
  - "Pat is a man or Pat is a woman" $\oplus$
- Need context to understand the meaning!
- For this class, assume "or" means inclusive

The Exclusive Or (cont.)

- The exclusive or of propositions p and q has this truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Connectives: *Implication*

- The conditional statement or implication $p \rightarrow q$ is the proposition "if p, then q"
- $p \rightarrow q$ is false when p is true and q is false, and is true otherwise
- p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)
- **Example:**
  - If p denotes "I am at home", and
  - q denotes "It is raining"
  - then $p \rightarrow q$ denotes "If I am at home then it is raining"

Implication (cont.)

- The implication or conditional statement $p \rightarrow q$ has this truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the hypothesis and the conclusion

- These implications are perfectly fine, but would not be used in ordinary English
  - "If the moon is made of green cheese, then I have more money than Bill Gates "
  - "If $1 + 1 = 3$, then pink elephants can fly"

Examples

- "If you get 100% on the final, then you will get an A"
- Interpretation:
  - If you manage to get a 100% on the final, then you would expect to receive an A
  - If you do not get a 100%, you may or may not receive an A, depending on other factors (such as…)
  - However, if you do get 100%, but the professor does not give you an A, you will feel cheated

More Examples

- “When I got elected, I will lower the taxes”
- Interpretation:
  - If the politician is elected, voters would expect the taxes to get lower
  - If the politician is not elected, then the voters have no expectations regarding the taxes, it might get lower, higher, or stay the same.
  - It is only when the politician is elected and the taxes are not lower, the voters would say that the politician has broken his campaign pledge

Different Ways of Expressing $p \rightarrow q$

- if p, then q
- if p, q
- p implies q
- p only if q
- a necessary condition for p is q
- p is sufficient for q
- q if p
- q whenever p
- q when p
- q follows from p
- q unless $\neg p$
- q is necessary for p
- a sufficient condition for q is p

Sufficient

- "if *p* then *q*" expresses the same thing as "*p* is sufficient for *q*"
  - This is basically saying that *p* holding is sufficient for concluding that *q* will also hold
- Example:
  - "if *p* then *q*": If Maria learns discrete mathematics, then she will find a good job
  - "*p* is sufficient for *q*": Learning discrete mathematics *is sufficient* for Maria to find a good job
    - If Maria doesn't learn discrete mathematics, she might find a good job or not, however if she does learn discrete mathematics, she will find a good job for sure

p	q	$p \to q$
F	F	T
F	T	T
T	**F**	**F**
T	T	T

Necessary

- "if p then q" expresses the same thing as "*q is necessary for p*"
  - This is basically saying that q holding is a necessary conclusion for a holding premise p
- Example:
  - "if p then q": If you can access the Internet, then you have paid a subscription fee
  - "q is necessary for p": Paying a subscription fee is necessary for being able to access the Internet
    - If you can access the Internet, then you must have paid a subscription fee, however if you cannot access the Internet, you may have paid a subscription fee or not

p	q	$p \to q$
F	F	T
F	T	T
T	**F**	**F**
T	T	T

Only If

- "if *p* then *q*" expresses the same thing as "*p* only if *q*"
  - This is basically saying that *p* cannot be true when *q* is not true
- Example:
  - "if *p* then *q*": If you can access the Internet, then you have paid a subscription fee"
  - "*p* only if *q*": You can access the Internet only if you pay a subscription fee
    - If you haven't paid a subscription fee, you cannot access the Internet, however if you did pay a subscription fee, you might be able to access the Internet or not

p	q	$p \to q$
F	F	T
F	T	T
T	**F**	**F**
T	T	T

Unless

- "if p then q" expresses the same thing as "q unless $\neg p$"
  - This is basically saying that q will be true except if p is false
- Example:
  - "if p then q": *If Maria learns discrete Mathematics, then she will find a good job*
  - "q *unless* $\neg p$": *Maria will find a good job unless she does not learn discrete mathematics*
    - We surely know Maria will get a good job, unless she doesn't learn discrete Mathematics – in that case, we don't know whether she'll find a good job or not

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	**F**	**F**
T	T	T

Converse, Contrapositive, and Inverse

- From $p \to q$, we can form new conditional statements
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$
- Out of these three conditional statements formed from $p \to q$, only the *contrapositive* always has the same truth value as $p \to q$
- The other two might seem to have a similar meaning, but actually translate to different premises and conclusions

Example

Let p denote "I am elected president", and

q denote "I will make healthcare free"

- *Original* $p \rightarrow q$ *:* "If I am elected president, I will make healthcare free"
- *Converse* $q \rightarrow p$: "If the healthcare was made free, then I am elected president"
- *Contrapositive* $\neg q \rightarrow \neg p$*:* "If healthcare wasn't made free, then I haven't been elected president"
- *Inverse* $\neg p \rightarrow \neg q$: "If I wasn't elected president, then the health care won't be made free"

Example Explained

- The contrapositive clearly means the same thing as the original statement
  - I made a pledge that healthcare will be made free when I get elected to presidency
  - Consequently, if the healthcare wasn't made free, then surely I haven't been elected for presidency
- The converse, on the other hand, means something else
  - It says that observing healthcare made for free will lead to concluding that I have been elected for presidency, which isn't what the original statement says
  - The original statement keeps space for the case when I don't get elected for presidency, yet someone else makes healthcare for free
- The same logic applies for the inverse

Exercise

- Find the converse, inverse, and contrapositive of “It raining is a sufficient condition for me not going to town”

- Solution:
  - Original : If it is raining, then I won’t go to town
  - Converse: If I do not go to town, then it is raining
  - Inverse: If it is not raining, then I will go to town
  - Contrapositive: If I go to town, then it is not raining

Another Exercise

- Find the converse, inverse, and contrapositive of "The home team wins whenever it is raining"

- Solution:
  - Original: if it is raining , then the home team wins
  - Converse: if the home team wins, then it is raining
  - Inverse: if it is not raining, the home team won't win
  - Contrapositive: If the home team doesn't win, then it is not raining

Connectives: *Biconditional*

- The biconditional proposition $p \leftrightarrow q$ is the proposition "p if and only if q"
  - The biconditional $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise
  - Also called bi-implications

- **Example**:
  - If p denotes "buying a ticket",
  - and q denotes "can take a flight"
  - then $p \leftrightarrow q$ denotes "I can take a flight *if and only if* I buy a ticket"
    - This statement is true when *you buy a ticket* and *can take a flight*, Or when *you don't buy a ticket* and *can't take a flight*
    - It is false when *you buy a ticket* and *can't take a flight* (like when the airlines bumps you)
    - Or when *you don't buy a ticket* and *can take a flight* (like when you win a ticket)

Biconditional

- The biconditional proposition $p \leftrightarrow q$ has this truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Expressing the Biconditional

- Some alternative ways "*p* if and only if *q*" is expressed in English
  - ***p*** **iff** ***q***
  - **if *p* then *q*, and conversely**
  - ***p*** **is necessary and sufficient for** ***q***
- Example:
  - You will be always informed if and only if you read a newspaper everyday
  - If you read a newspaper everyday, then you will be informed, and conversely
  - To be always informed, it is necessary and sufficient to read a newspaper everyday

Implicit Use Of Biconditionals

- Human natural language is not explicit on biconditionals most of the times
- People use "if, then" and "only if " constructs and mean "if and only if " implicitly
- For example, consider the statement in English: "If you finish your meal, then you can have dessert"
- What is really meant is "You can have dessert if and only if you finish your meal"
- Here, we will always distinguish between the conditional statement $p \rightarrow q$ and the biconditional statement $p \leftrightarrow q$

Precedence of Logical Operators

- When multiple logical operators are used in the same compound proposition, we follow the following precedence order to understand it:
  - Note that all operators are right associative *(conventionally)*
  - Parentheses can also be used to disambiguate

- Example:
  - $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
  - If the intended meaning is $p \vee (q \rightarrow \neg r)$, then parentheses must be used

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Another Example

- Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

Op.	Prec.
$\neg$	1
$\land$ $\lor$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \wedge \neg q \rightarrow p \wedge r \vee s \rightarrow \neg t \leftrightarrow q \vee s$$

$$p \wedge (\neg q) \rightarrow p \wedge r \vee s \rightarrow (\neg t) \leftrightarrow q \vee s$$

Op.	Prec.
¬	1
∧ ∨	2 3
→ ↔	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

Op.	Prec.
¬	1
∧ ∨	2 3
→ ↔	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \leftrightarrow (q \lor s)$$

Op.	Prec.
¬	1
∧ ∨	2 3
→ ↔	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \leftrightarrow (q \lor s)$$

$$(p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t)) \leftrightarrow (q \lor s)$$

Op.	Prec.
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \wedge \neg q \rightarrow p \wedge r \vee s \rightarrow \neg t \leftrightarrow q \vee s$$

$$p \wedge (\neg q) \rightarrow p \wedge r \vee s \rightarrow (\neg t) \leftrightarrow q \vee s$$

$$(p \wedge (\neg q)) \rightarrow (p \wedge r) \vee s \rightarrow (\neg t) \leftrightarrow q \vee s$$

$$(p \wedge (\neg q)) \rightarrow ((p \wedge r) \vee s) \rightarrow (\neg t) \leftrightarrow (q \vee s)$$

$$(p \wedge (\neg q)) \rightarrow (((p \wedge r) \vee s) \rightarrow (\neg t)) \leftrightarrow (q \vee s)$$

$$((p \wedge (\neg q)) \rightarrow (((p \wedge r) \vee s) \rightarrow (\neg t))) \leftrightarrow (q \vee s)$$

Op.	Prec.
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Another Example

- Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$
$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$
$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \leftrightarrow q \lor s$$
$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \leftrightarrow (q \lor s)$$
$$(p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t)) \leftrightarrow (q \lor s)$$
$$((p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t))) \leftrightarrow (q \lor s)$$
$$(((p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t))) \leftrightarrow (q \lor s))$$

Op.	Prec.
¬	1
∧ ∨	2 3
→ ↔	4 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Truth Tables

Truth Tables For Compound Propositions

- Rows
  - Need a row for every possible combination of values for the atomic propositions
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬ r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value
- Example:
  - Show using a truth table that the implication is equivalent to the contrapositive: $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
  - Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

- Example:
  - Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication
- Solution:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Biconditional Truth Table

- $p \leftrightarrow q$ means that p and q have the **same** truth value
- Note this truth table is the exact opposite of ⊕'s!

  Thus, $p \leftrightarrow q$ means $\neg(p \oplus q)$
- $p \leftrightarrow q$ does not imply that p and q are true, or that either of them causes the other, or that they have a common cause

p	q	p⊕q	¬(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Exercise

- How many rows are there in a truth table with *n* propositional variables?
- Solution: 2^n
- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions

Semantics of propositional formulae

- A proposition is valid iff it evaluates to true in every row of its truth table
- A proposition is satisfiable iff it evaluates to true in some row of its truth table
- A proposition is valid iff its negation is not satisfiable
- A proposition is a contingency iff it is satisfiable and its negation is satisfiable
- A proposition is a contradiction iff it evalutes to false in every row of its truth table, i.e., iff its negation is valid

Any Questions?