

### **Discrete Structures**

Topic 1 – Logic: Propositional Logic (Ch 1.1)\*

CMPS 211 – Fall 2017 – American University of Beirut

\* Extracted from Discrete Mathematics and It's Applications book slides

# Why logic?

- Logic is a set of principles that can be used to reason about (mathematical) statements
- For instance, let's say we want to formally express and reason about the following statement:
  - For every positive integer *n*, the sum of the positive integers not exceeding *n* is n(n+1)/2"
- We can formally express the above statement using logic
- We can also prove the above statement or argue whether it is true or false using logic

## Why logic? (cont.)

- Logic has numerous applications to Computer Science
  - Used in the design of computer circuits
  - Used in the construction of computer programs
  - Used to verify the correctness of programs
  - Used to ensure the security of a system
  - Used heavily in artificial intelligence

# **Propositional Logic**

## Propositions

- A proposition is a declarative statement that is either true or false
- Examples of propositions:
  - a) The Moon is made of white cheese
  - b) Toronto is the capital of Canada
  - c) A week has more days than a month
  - d) 1 + 0 = 1
  - e) 0 + 0 = 2
- Examples that are not proposition
  - a) Sit down! *command*
  - b) What time is it? *question*
  - c) **1 + 2** *expressions with a non-true/false value*
  - d) This statement is false *a paradox*

## Atomic propositions

- We use letters (p, q, r, s, ...) to denote atomic propositions
  - Also called propositional variables
  - Similar to x, y, z, ... for numerical variables
  - For example, let p be the proposition that the earth is round and q be the proposition that the moon is flat
  - These represent a single statement that cannot be "decomposed"

## **Compound propositions**

- Compound propositions are built up from atomic propositions by the use of Boolean connectives. Also called propositional formulae.
- Propositional Logic is the logic of propositional formulae and their meaning
  - First developed by the Greek philosopher Aristotle more than 2300 years ago
  - George Boole introduced Boolean Algebra in 1854

## **Propositional Connectives**

## Propositional Operators/Connectives

- An operator or connective combines one or more operand expressions into a larger expression (*e.g.*, "+" in numeric expressions)
  - ▶ *Unary* operators take 1 operand (*e.g.*, −3)
  - binary operators take 2 operands (e.g., 3 × 4)
- Propositional or Boolean operators operate on propositions (or their truth values) instead of numbers
- There are six main operators
  - ▶ Negation ¬
  - Conjunction  $\wedge$
  - Disjunction V
  - The Exclusive Or  $\oplus$
  - Implication  $\rightarrow$
  - Biconditional  $\leftrightarrow$

## Connectives: Negation

- ▶ The negation of a proposition *p* is denoted by ¬*p*
- In an English statement, we express ¬p as follows: "It's not the case that p"
  - ▶ ¬p is true if p is false and is false if p is true
- Example:
  - If p denotes "I am at home",
  - then ¬p denotes "It is not the case that I am at home" or more simply "I am not at home"
- The negation of a proposition p has this truth table

| p | $\neg p$ |
|---|----------|
| Т | F        |
| F | Т        |

# Connectives: Conjunction

- The conjunction of propositions *p* and *q* is denoted by *p*∧ *q*
- It is the proposition "p and q"
- *p* ∧ *q* is true when both *p* and *q* are true and is false otherwise
- Example:
  - If p denotes "I am at home", and
  - q denotes "It is raining"
  - then  $p \wedge q$  denotes "I am at home and it is raining"

Conjunction (cont.)

The conjunction of propositions p and q has this truth table

| p | q | $p \wedge q$ |
|---|---|--------------|
| Т | Т | Т            |
| Т | F | F            |
| F | Т | F            |
| F | F | F            |

Connectives: Disjunction

- The disjunction of propositions *p* and *q* is denoted by *p*∨*q*
- It is the proposition "p or q"
- ▶ p∨q is false when both p and q are false and is true otherwise

• Example:

- If p denotes "I am at home", and
- q denotes "It is raining"
- ▶ then p∨q denotes "I am at home or it is raining"

Disjunction (cont.)

The disjunction of propositions p and q has this truth table

| p | q | $p \lor q$ |
|---|---|------------|
| Т | Т | Т          |
| Т | F | Т          |
| F | Т | Т          |
| F | F | F          |

## Connectives: The Exclusive Or

- The exclusive or of p and q is denoted by  $p \oplus q$
- We say *p XOR q*
- ▶ p ⊕ q is true when exactly either p or q is true and is false otherwise

### • Example:

- If p denotes "I am at home", and
- q denotes "It is raining"
- ▶ then p ⊕ q denotes :"either I am at home or it is raining" but not both

## The Exclusive Or (cont.)

- Note that English "or" can be ambiguous regarding the "both" case!
  - "Pat is a singer or Pat is a writer"
  - "Pat is a man or Pat is a woman"
- Need context to understand the meaning!
- For this class, assume "or" means inclusive

## The Exclusive Or (cont.)

The exclusive or of propositions p and q has this truth table

| p | q | $p \oplus q$ |
|---|---|--------------|
| Т | Т | F            |
| Т | F | Т            |
| F | Т | Т            |
| F | F | F            |

## Connectives: Implication

- The conditional statement or implication p→q is the proposition "if p, then q"
- *p*→*q* is false when *p* is true and *q* is false, and is true otherwise
- *p* is called the hypothesis (or antecedent or premise) and *q* is called the conclusion (or consequence)
- Example:
  - ▶ If *p* denotes "I am at home", and
  - q denotes "It is raining"
  - then  $p \rightarrow q$  denotes "If I am at home then it is raining"

Implication (cont.)

The implication or conditional statement p →q has this truth table

| p | q | $p \rightarrow q$ |
|---|---|-------------------|
| Т | Т | Т                 |
| Т | F | F                 |
| F | Т | Т                 |
| F | F | Т                 |

## **Understanding Implication**

- In p →q there does not need to be any connection between the hypothesis and the conclusion
- These implications are perfectly fine, but would not be used in ordinary English
  - "If the moon is made of green cheese, then I have more money than Bill Gates"
  - "If 1 + 1 = 3, then pink elephants can fly"

## Examples

"If you get 100% on the final, then you will get an A"

### Interpretation:

- If you manage to get a 100% on the final, then you would expect to receive an A
- If you do not get a 100%, you may or may not receive an A, depending on other factors (such as...)
- However, if you do get 100%, but the professor does not give you an A, you will feel cheated

## More Examples

- "When I got elected, I will lower the taxes"
- Interpretation:
  - If the politician is elected, voters would expect the taxes to get lower
  - If the politician is not elected, then the voters have no expectations regarding the taxes, it might get lower, higher, or stay the same.
  - It is only when the politician is elected and the taxes are not lower, the voters would say that the politician has broken his campaign pledge

# Different Ways of Expressing $p \rightarrow q$

- ▶ if *p*, then *q*
- ▶ if *p*, *q*
- *p* implies *q*
- *p* only if *q*
- a necessary condition
   for *p* is *q*
- *p* is sufficient for *q*

- ▶ *q* if *p*
- q whenever p
- ▶ *q* when *p*
- *q* follows from *p*
- ▶ q unless ¬p
- *q* is necessary for *p*
- a sufficient condition for q is p

# Sufficient

- "if p then q" expresses the same thing as "p is sufficient for q"
  - This is basically saying that p holding is sufficient for concluding that q will also hold

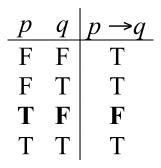
• Example:

- "if p then q": If Maria learns discrete mathematics, then she will find a good job
- "*p* is sufficient for *q*": Learning discrete mathematics *is sufficient* for Maria to find a good job
  - If Maria doesn't learn discrete mathematics, she might find a good job or not, however if she does learn discrete mathematics, she will find a good job for sure

 $\begin{array}{c|ccc} p & q & p \rightarrow q \\ \hline F & F & T \\ F & T & T \\ \hline T & F & F \\ T & T & T \end{array}$ 

### Necessary

- "if p then q" expresses the same thing as "q is necessary for p"
  - This is basically saying that q holding is a necessary conclusion for a holding premise p
- Example:
  - "if p then q": If you can access the Internet, then you have paid a subscription fee
  - "q is necessary for p": Paying a subscription fee is necessary for being able to access the Internet
    - If you can access the Internet, then you must have paid a subscription fee, however if you cannot access the Internet, you may have paid a subscription fee or not

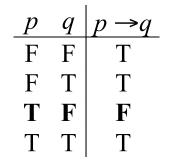


# Only If

- "if p then q" expresses the same thing as "p only if q"
  - This is basically saying that p cannot be true when q is not true

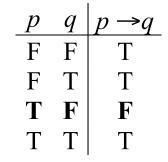
## Example:

- "if p then q": If you can access the Internet, then you have paid a subscription fee"
- "p only if q": You can access the Internet only if you pay a subscription fee
  - If you haven't paid a subscription fee, you cannot access the Internet, however if you did pay a subscription fee, you might be able to access the Internet or not



### Unless

- ▶ "if p then q" expresses the same thing as "q unless ¬p"
  - This is basically saying that q will be true except if p is false
- Example:
  - "if p then q": If Maria learns discrete Mathematics, then she will find a good job
  - "q unless ¬p": Maria will find a good job unless she does not learn discrete mathematics
    - We surely know Maria will get a good job, unless she doesn't learn discrete Mathematics – in that case, we don't know whether she'll find a good job or not



## Converse, Contrapositive, and Inverse

### From $p \rightarrow q$ , we can form new conditional statements

- $q \rightarrow p$  is the **converse** of  $p \rightarrow q$
- $\neg q \rightarrow \neg p$  is the **contrapositive** of  $p \rightarrow q$
- ▶ ¬  $p \rightarrow$  ¬ q is the **inverse** of  $p \rightarrow q$
- Out of these three conditional statements formed from *p* → *q*, only the *contrapositive* always has the same truth value as *p* → *q*
- The other two might seem to have a similar meaning, but actually translate to different premises and conclusions

## Example

Let *p* denote "I am elected president", and *q* denote "I will make healthcare free"

- Original p→q : "If I am elected president, I will make healthcare free"
- Converse q →p: "If the healthcare was made free, then I am elected president"
- Contrapositive ¬q → ¬p: "If healthcare wasn't made free, then I haven't been elected president"
- Inverse ¬p→ ¬q: "If I wasn't elected president, then the health care won't be made free"

# Example Explained

- The contrapositive clearly means the same thing as the original statement
  - I made a pledge that healthcare will be made free when I get elected to presidency
  - Consequently, if the healthcare wasn't made free, then surely I haven't been elected for presidency

#### The converse, on the other hand, means something else

- It says that observing healthcare made for free will lead to concluding that I have been elected for presidency, which isn't what the original statement says
- The original statement keeps space for the case when I don't get elected for presidency, yet someone else makes healthcare for free
- The same logic applies for the inverse

### Exercise

- Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for me not going to town"
- Solution:
  - Original : If it is raining, then I won't go to town
  - Converse: If I do not go to town, then it is raining
  - Inverse: If it is not raining, then I will go to town
  - Contrapositive: If I go to town, then it is not raining

# Another Exercise

- Find the converse, inverse, and contrapositive of "The home team wins whenever it is raining"
- Solution:
  - Original: if it is raining , then the home team wins
  - Converse: if the home team wins, then it is raining
  - Inverse: if it is not raining, the home team won't win
  - Contrapositive: If the home team doesn't win, then it is not raining

## Connectives: Biconditional

- The biconditional proposition p ↔ q is the proposition "p if and only if q"
  - The biconditional p ↔ q is true when p and q have the same truth values and is false otherwise
  - Also called bi-implications

### • Example:

- If p denotes "buying a ticket",
- and q denotes "can take a flight"
- then  $p \leftrightarrow q$  denotes "I can take a flight *if and only if* I buy a ticket"
  - This statement is true when you buy a ticket and can take a flight, Or when you don't buy a ticket and can't take a flight
  - It is false when you buy a ticket and can't take a flight (like when the airlines bumps you)
  - Or when you don't buy a ticket and can take a flight (like when you win a ticket)

## Biconditional

• The biconditional proposition  $p \leftrightarrow q$  has this truth table

| р | q | $p \leftrightarrow q$ |
|---|---|-----------------------|
| Т | Т | Т                     |
| Т | F | F                     |
| F | Т | F                     |
| F | F | Т                     |

## Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English
  - p iff q
  - if p then q, and conversely
  - ▶ *p* is necessary and sufficient for *q*
- Example:
  - You will be always informed if and only if you read a newspaper everyday
  - If you read a newspaper everyday, then you will be informed, and conversely
  - To be always informed, it is necessary and sufficient to read a newspaper everyday

## Implicit Use Of Biconditionals

- Human natural language is not explicit on biconditionals most of the times
- People use "if, then" and "only if " constructs and mean "if and only if " implicitly
- For example, consider the statement in English: "If you finish your meal, then you can have dessert"
- What is really meant is "You can have dessert if and only if you finish your meal"
- Here, we will always distinguish between the conditional statement *p* → *q* and the biconditional statement *p* ↔ *q*

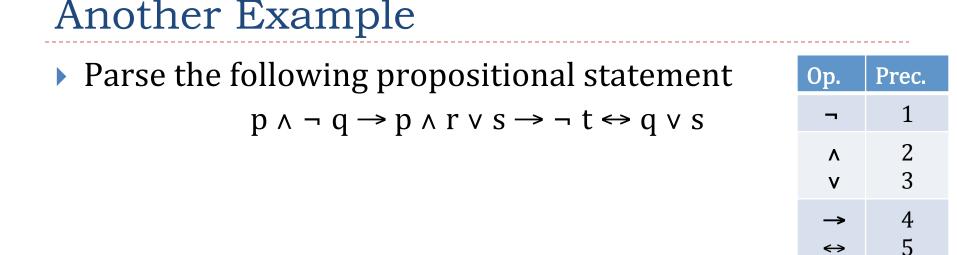
# Precedence of Logical Operators

- When multiple logical operators are used in the same compound proposition, we follow the following precedence order to understand it:
  - Note that all operators are right associative (conventionally)
  - Parentheses can also be used to disambiguate

### • Example:

- ▶  $p \lor q \rightarrow \neg r$  is equivalent to  $(p \lor q) \rightarrow \neg r$
- If the intended meaning is pv(q→¬r), then parentheses must be used

| Operator          | Precedence |
|-------------------|------------|
| -                 | 1          |
| ٨                 | 2          |
| V                 | 3          |
| $\rightarrow$     | 4          |
| $\Leftrightarrow$ | 5          |

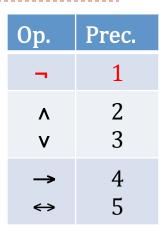


- Note that all operators are right associative
- Parentheses can also be used to disambiguate

# Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$



- Note that all operators are right associative
- Parentheses can also be used to disambiguate

#### Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

Op.Prec.
$$\neg$$
1 $\land$ 2 $\lor$ 3 $\rightarrow$ 4 $\leftrightarrow$ 5

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \Leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

 $(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \Leftrightarrow (q \lor s)$ 

| Op.      | Prec. |
|----------|-------|
| -        | 1     |
| ∧        | 2     |
| ∨        | 3     |
| <b>→</b> | 4     |
| ↔        | 5     |

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

#### Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \leftrightarrow q \lor s$$

Prec.

1

2

3

5

**Op.** 

Λ

V

 $\Leftrightarrow$ 

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \Leftrightarrow (q \lor s)$$
$$(p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t)) \Leftrightarrow (q \lor s)$$

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

#### Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \Leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$
$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \Leftrightarrow (q \lor s)$$

Dp.Prec.
$$\neg$$
1 $\land$ 2 $\lor$ 3 $\rightarrow$ 4 $\leftrightarrow$ 5

 $(p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t)) \Leftrightarrow (q \lor s)$  $((p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t))) \Leftrightarrow (q \lor s)$ 

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

 $(p \land (\neg q)) \rightarrow (((p \land q)))$ 

 $((p \land (\neg q)) \rightarrow (((p \land q))))$ 

Parse the following propositional statement

$$p \land \neg q \rightarrow p \land r \lor s \rightarrow \neg t \Leftrightarrow q \lor s$$

$$p \land (\neg q) \rightarrow p \land r \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$

$$(p \land (\neg q)) \rightarrow (p \land r) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s$$
$$(p \land (\neg q)) \rightarrow ((p \land r) \lor s) \rightarrow (\neg t) \Leftrightarrow (q \lor s)$$

$$(\neg t) \lor s \rightarrow (\neg t) \Leftrightarrow q \lor s \rightarrow (\neg t) \lor q \lor s \rightarrow (\neg t) \lor (q \lor s)$$

$$\wedge r) \lor s) \rightarrow (\neg t)) \Leftrightarrow (q \lor s)$$

$$\wedge r) \lor s) \rightarrow (\neg t)) \Leftrightarrow (q \lor s)$$

Prec.

1

2

3

4

5

**Op.** 

Λ

V

 $(((p \land (\neg q)) \rightarrow (((p \land r) \lor s) \rightarrow (\neg t))) \Leftrightarrow (q \lor s))$ 

- Note that all operators are right associative
- Parentheses can also be used to disambiguate

### Truth Tables

## Truth Tables For Compound Propositions

#### Rows

- Need a row for every possible combination of values for the atomic propositions
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up
    - This includes the atomic propositions

# Example Truth Table

 $\blacktriangleright$  Construct a truth table for  $p \lor q \to \neg r$ 

| р | q | r | ¬r | p v q | $p \lor q \rightarrow \neg r$ |
|---|---|---|----|-------|-------------------------------|
| Т | Т | Т | F  | Т     | F                             |
| Т | Т | F | Т  | Т     | Т                             |
| Т | F | Т | F  | Т     | F                             |
| Т | F | F | Т  | Т     | Т                             |
| F | Т | Т | F  | Т     | F                             |
| F | Т | F | Т  | Т     | Т                             |
| F | F | Т | F  | F     | Т                             |
| F | F | F | Т  | F     | Т                             |

# **Equivalent Propositions**

- Two propositions are *equivalent* if they always have the same truth value
- Example:
  - Show using a truth table that the implication is equivalent to the contrapositive: (p→q) ⇔ (¬q → ¬p)
  - Solution:

| p | q | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ |
|---|---|----------|----------|-------------------|-----------------------------|
| Т | Т | F        | F        | Т                 | Т                           |
| Т | F | F        | Т        | F                 | F                           |
| F | Т | Т        | F        | Т                 | Т                           |
| F | F | Т        | Т        | Т                 | Т                           |

### Using a Truth Table to Show Non-Equivalence

### • Example:

Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication

### Solution:

| p | q | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $\neg p \rightarrow \neg q$ | $q \rightarrow p$ |
|---|---|----------|----------|-------------------|-----------------------------|-------------------|
| Т | Т | F        | F        | Т                 | Т                           | Т                 |
| Т | F | F        | Т        | F                 | Т                           | Т                 |
| F | Т | Т        | F        | Т                 | F                           | F                 |
| F | F | Т        | Т        | Т                 | Т                           | Т                 |

# **Biconditional Truth Table**

- ▶  $p \leftrightarrow q$  means that p and q have the same truth value
- Note this truth table is the exact opposite of ⊕'s!
   Thus, p ↔ q means ¬(p ⊕ q)
- p ↔ q does not imply that p and q are true, or that either of them causes the other, or that they have a common cause

| р | q | p⊕q | ¬(p⊕q) | p⇔q |
|---|---|-----|--------|-----|
| Т | Т | F   | Т      | Т   |
| Т | F | Т   | F      | F   |
| F | Т | Т   | F      | F   |
| F | F | F   | Т      | Т   |

## Exercise

- How many rows are there in a truth table with n propositional variables?
- ▶ Solution: 2<sup>n</sup>
- Note that this means that with n propositional variables, we can construct 2<sup>n</sup> distinct (i.e., not equivalent) propositions

# Semantics of propositional formulae

- A proposition is valid iff it evaluates to true in every row of its truth table
- A proposition is satisfiable iff it evaluates to true in some row of its truth table
- A proposition is valid iff its negation is not satisfiable
- A proposition is a contingency iff it is satisfiable and its negation is satisfiable
- A proposition is a contradiction iff it evalutes to false in every row of its truth table, i.e., iff its negation is valid

# Any Questions?