Problem 1 (20 Points)

a. (10) Using truth tables, show that

$(p \to r) \wedge (q \to r)$ is equivalent to $(p \vee q) \to r$

~~Use truth tables to show that the given compound proposition is a tautology.~~

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

b. (10) Using (a), and without using truth tables, show that the following is a tautology, using equivalences. Justify each step in your proof.

$$(p \vee q) \wedge (p \to r) \wedge (q \to r) \ \to \ r$$

$$\equiv (p \vee q) \wedge ((p \vee q) \to r) \ \to \ r \quad \text{(part (a))}$$

$$\equiv \text{T} \quad \text{(Modus Ponens: } P \wedge (P \to r) \ \to \ r \text{ is a tautology with } P \text{ being } p \vee q\text{)}$$

Can also be done by using $p \to q \equiv \neg p \vee q$ several times, and then simplifying the expression to show that it is equivalent to T.

Problem 2 (15 Points)

Consider the following predicates:

$S(x)$: "x is a student in this class",
$J(x)$: "x is a student who knows how to write Java programs",
$C(x)$: "x is a student who knows how to write C++ programs".
$P(x)$: "x is a student who can get a well-paying job".

Where x is a variable whose domain is the set of students in the university.
Express each of the following in predicate logic:

a. (5) Salwa is a student who is not in this class, but knows how to program in C++.

$\neg S(Salwa) \wedge C(Salwa)$

b. (5) There are students in this class that do not know how to program in both Java and C++

$\exists x\, (S(x) \wedge(\neg J(x) \vee \neg C(x))$

OR

$\exists x\, (S(x) \wedge \neg(J (x) \wedge C(x)))$

Another answer which reflects another meaning of the sentence (English is ambiguous) is

$\exists x\, (S(x) \wedge \neg J(x) \wedge \neg C(x))$

Both answers have been considered correct.

c. (5) There are at least two students in this class who know how to program in Java

$\exists x\, \exists y\, (S(x) \wedge S (y) \wedge J(x) \wedge J(y) \wedge J(x) \wedge \neg(x=y))$

Problem 3 (15 Points)

Consider the following argument involving the predicates above of the previous problem:

$$\forall x\,(\ S(x) \rightarrow J(x) \vee C(x)\ )$$
$$S(Amin) \wedge \neg C(Amin)$$
$$\forall x\,(\ J(x) \rightarrow P(x)\ )$$

$$\therefore\ \exists x\,(\ S(x) \wedge P(x)\ )$$

a. (10) Establish the validity of the argument justifying each step.

1.	$\forall x\,(\ S(x) \rightarrow J(x) \vee C(x)\ )$	Premise
2.	$S(Amin) \rightarrow J(Amin) \vee C(Amin)$	(1) and Universal Instantiation (UI)
3.	$S(Amin) \wedge \neg C(Amin)$	Premise
4.	$S(Amin)$	(4), Simplification
5.	$J(Amin) \vee C(Amin)$	(2), (4) and Modus Ponens
6.	$\neg C(Amin) \wedge S(Amin)$	(3), $\wedge$ is Commutative
7.	$\neg C(Amin)$	(6), Simplification
8.	$C(Amin) \vee J(Amin)$	(5), $\vee$ is Commutative
9.	$J(Amin)$	(8), (7), Disjunctive Syllogism
10.	$S(Amin) \wedge J(Amin)$	(4), (9) and Conjunction
11.	$\therefore\ \exists x\,(\ S(x) \wedge P(x)\ )$	(10), Existential Generalization

b. (5) Express in English the argument above.

Any student in the class knows how to program in Java or in C++. Amin is a student in the class who does not know how to program in C++. Any student who knows how to program in Java can get a well-paid job. Therefore, some student in the class can get a well-paid job.

Problem 4 (15 Points)

Recall that a real number x is **rational** if it can be written as the quotient of two integers; i.e. $x = m/n$, where m and n are integers, n non-zero; Otherwise, x is said to be **irrational**.

Prove or disprove each of the following. In each case specify which method of proof did you use.

a. (5) The product of two rational numbers is rational.

Direct Proof:

Let x and y be any two rational numbers.
Then $x = \frac{m}{n}$ and $y = \frac{p}{q}$ for some integers $m, n, n \neq 0$, and $p, q, q \neq 0$.
So $xy = = \frac{mp}{nq}$. mp and nq are integers, with $nq \neq 0$ (since $n \neq 0$, and $q \neq 0$)
Therefore xy is rational.

b. (5) The sum of two irrational numbers is irrational.

Counter Example:

Let $x = \sqrt{2}$, and $y = -\sqrt{2}$. Then x and y are irrational. Yet their sum is 0, $x + y = \sqrt{2} + (-\sqrt{2}) = 0$.
And 0 is rational.

c. (5) The sum of a rational number and an irrational number is irrational.

Proof by Contradiction:

Let x be a rational number, $x = \frac{m}{n}$ with integers $m, n, n \neq 0$, and let y irrational. Suppose that $x+y=z$.
For the sake of contradiction, assume z is rational. Then $z = \frac{p}{q}$ with integers p $, q, q \neq 0$.
Thus $y = z - x = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq}$
But $mq-np$ is an integer, and nq is an integer, $nq \neq 0$ (since $n \neq 0$, and $q \neq 0$)

So y is rational.

Contradiction (y is irrational and y is rational).

So our assumption that z is rational is false. So z is irrational.

Problem 5. (10 Points)

Suppose $B = \{ 2, \{2\} \}$.   Mark each of the following statements as TRUE or FALSE (Circle the right answer)

a.	$\{2\} \in B$.	TRUE	FALSE
b.	$\{2\} \subseteq B$.	TRUE	FALSE
c.	$2 \subseteq B$.	TRUE	FALSE
d.	$\mid B \mid = 1$	TRUE	FALSE
e.	$\phi = \{\phi\}$	TRUE	FALSE

Problem 6. (10 Points)

Suppose that the set membership table for a set expression *X* involving two sets *A* and *B* is the following:

A	*B*	X
1	1	0
1	0	1
0	1	0
0	0	1

Give a set expression for *X*, in terms of *A* and *B* (or their complements!)

Similar to Distributive Normal Form adapted to Set Membership. Then have a union of intersections: See where there is a 1 in the column of *X*; then form the intersection of *A* or its complement with *B* or its complement, depending on whether *A*=1 (then include *A*) or 0 (then include *A's* complement), and the same for *B*.

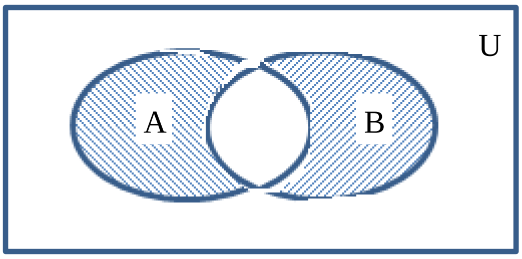
So here: $X = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$

Problem 7 (25 Points)

The symmetric difference of two sets A and B denoted by $A \oplus B$ is the set of elements that are in A or in B but not in both!! So $\{1, 3, 5\} \oplus \{1, 2, 3\} = \{2, 5\}$.

$$x \in A \oplus B \text{ is equivalent to } x \in (A \cup B) \wedge x \notin (A \cap B)$$

a. (5) Draw a Venn diagram to represent $A \oplus B$.



b. (10) Use set builder notation and propositional logic to show that $A \oplus B = (A\text{-}B) \cup (B\text{-}A)$

$A \oplus B = \{ x \mid x \in (A \cup B) \wedge x \notin (A \cap B) \}$ by definition

$= \{ x \mid (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B) \}$

$= \{ x \mid (x \in A \vee x \in B) \wedge (\neg x \in A \vee \neg x \in B) \}$ De Morgans

$= \{ x \mid ((x \in A \vee x \in B) \wedge \neg x \in A) \vee ((x \in A \vee x \in B) \wedge \neg x \in B) \}$ Distributive property of $\wedge$ wrt $\vee$

$= \{ x \mid ((x \in A \wedge \neg x \in A) \vee (x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in B)) \}$

Distributive property of $\wedge$ wrt $\vee$

$= \{ x \mid (F \vee (x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B) \vee F) \}$ Negation laws $p \wedge \neg p \equiv F$

$= \{ x \mid ((x \in B \wedge \neg x \in A)) \vee ((x \in A \wedge \neg x \in B)) \}$ Identity law $p \vee F \equiv p$

$= \{ x \mid ((x \in B\text{-}A)) \vee ((x \in A\text{-}B)) \}$ Definition of $B\text{-}A$ and $A\text{-}B$

$= (B\text{-}A) \cup (A\text{-}B)$ Definition of $\cup$

$= (A\text{-}B) \cup (B\text{-}A)$ $\cup$ is commutative

c. (10) Show that $A = B$ if and only if $A \oplus B = \phi$. (Hint: Use (b), and if $A\text{-}B = \phi$ then $A \subseteq B$)

Suppose $A \oplus B = \phi$, then $(A\text{-}B) \cup (B\text{-}A) = \phi$. So $A\text{-}B = \phi$ and $A\text{-}B = \phi$. So $A \subseteq B$ and $B \subseteq A$. So $A = B$.

Now, suppose $A = B$. Then $A\text{-}B = \phi$ and $A\text{-}B = \phi$. So $(A\text{-}B) \cup (B\text{-}A) = \phi$. Therefore $A \oplus B = \phi$.