American University of Beirut
Dept. of Computer Science

CMPS/Math 211

Discrete Structures

Lecture 3

Section 1.2: Propositional Equivalences

Propositional Equivalence (§1.2)

Two *syntactically* (*i.e.,* textually) different compound propositions may be the *semantically* identical (*i.e.,* have the same meaning). We call them *equivalent*. Learn:

- Various *equivalence rules* or *laws*.
- How to *prove* equivalences using *symbolic derivations*.

Tautologies and Contradictions

A *tautology* is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!

Ex. $p \vee \neg p$ [What is its truth table?]

A *contradiction* is a compound proposition that is **false** no matter what! *Ex.* $p \wedge \neg p$ [Truth table?]

Other compound props. are *contingencies*.

Logical Equivalence

Compound proposition p is *logically equivalent* to compound proposition q, written $p \Leftrightarrow q$, **IFF** the compound proposition $p \leftrightarrow q$ is a tautology.

Compound propositions p and q are logically equivalent to each other **IFF** p and q contain the same truth values as each other in <u>all</u> rows of their truth tables.

Proving Equivalence via Truth Tables

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match all or part of a much more complicated proposition and to find an equivalence for it.

Equivalence Laws - Examples

- *Identity*: $p \wedge \mathbf{T} \Leftrightarrow p$ $\quad p \vee \mathbf{F} \Leftrightarrow p$
- *Domination*: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$ $\quad p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- *Idempotent*: $p \vee p \Leftrightarrow p$ $\quad p \wedge p \Leftrightarrow p$
- *Double negation*: $\neg\neg p \Leftrightarrow p$
- *Commutative*: $p \vee q \Leftrightarrow q \vee p$ $\quad p \wedge q \Leftrightarrow q \wedge p$
- *Associative*: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p\vee(q\wedge r) \Leftrightarrow (p\vee q)\wedge(p\vee r)$
  $p\wedge(q\vee r) \Leftrightarrow (p\wedge q)\vee(p\wedge r)$
- *De Morgan's*:
  $\neg(p\wedge q) \Leftrightarrow \neg p \vee \neg q$
  $\neg(p\vee q) \Leftrightarrow \neg p \wedge \neg q$
- *Trivial tautology/contradiction*:
  $p \vee \neg p \Leftrightarrow \mathbf{T}$ $\quad$ $p \wedge \neg p \Leftrightarrow \mathbf{F}$

!!!

- Consult your textbook for arbitrarily sized conjunctions, disjunctions, and laws applicable to them.
- Example: Use De Morgan's laws to express the negations of ``Miguel has a cellphone and he has a laptop computer'' and ``Heather will go to the concert or Steve will go to the concert''

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p\oplus q \Leftrightarrow (p\vee q)\wedge\neg(p\wedge q)$
  $p\oplus q \Leftrightarrow (p\wedge\neg q)\vee(q\wedge\neg p)$
- Implies: $p\rightarrow q \Leftrightarrow \neg p \vee q$
- Biconditional: $p\leftrightarrow q \Leftrightarrow (p\rightarrow q) \wedge (q\rightarrow p)$
  $p\leftrightarrow q \Leftrightarrow \neg(p\oplus q)$

Proving Equivalences

- We can prove equivalences using:
  - Truth tables.
  - Symbolic derivations. $p \Leftrightarrow q \Leftrightarrow r$ ...

Prove Equivalences using a Truth Table

- Examples:

1- Show that $\neg (p \lor q)$ and $\neg p \land \neg q$ are logically equivalent

2- Show that $p \rightarrow q$ and $\neg p \lor q$ are logically equivalent

3- Show that $p \lor (q \land r)$ and

$(p \lor q) \land (p \lor r)$ are logically equivalent

Prove Equivalences using Symbolic Derivation

- Show that $\neg (p \rightarrow q)$ and $p \land \neg q$ are equivalent.

Prove Equivalences using Symbolic Derivation

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$\neg (p \rightarrow q) \Leftrightarrow \neg(\neg p \lor q)$ (shown earlier)

Prove Equivalences using Symbolic Derivation

- Show that $\neg (p \rightarrow q)$ and $p \land \neg q$ are equivalent.

$\neg (p \rightarrow q) \Leftrightarrow \neg(\neg p \lor q)$ (shown earlier)

$\Leftrightarrow \neg(\neg p) \land \neg q$ (De Morgan's)

Prove Equivalences using Symbolic Derivation

- Show that $\neg (p \to q)$ and $p \wedge \neg q$ are equivalent.

$\neg (p \to q) \Leftrightarrow \neg(\neg p \vee q)$ (shown earlier)

$\Leftrightarrow \neg(\neg p) \wedge \neg q$ (De Morgan's)

$\Leftrightarrow p \wedge \neg q$ (Double Negation)

Prove Equivalences using Symbolic Derivation

- Show that $(p \wedge q) \to (p \vee q)$ is a tautology

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$(p \wedge q) \to (p \vee q) \Leftrightarrow \neg(p \wedge q) \vee (p \vee q)$

Prove Equivalences using Symbolic Derivation

- Show that $(p \wedge q) \to (p \vee q)$ is a tautology

$(p \wedge q) \to (p \vee q) \Leftrightarrow \neg(p \wedge q) \vee (p \vee q)$

$\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)$

Prove Equivalences using Symbolic Derivation

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

$$\begin{aligned}(p \wedge q)\rightarrow(p \vee q) &\Leftrightarrow \neg(p \wedge q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee p) \vee (\neg q \vee q)\end{aligned}$$

Prove Equivalences using Symbolic Derivation

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

$$\begin{aligned}(p \wedge q)\rightarrow(p \vee q) &\Leftrightarrow \neg(p \wedge q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee p) \vee (\neg q \vee q)\\ &\Leftrightarrow T \vee T\end{aligned}$$

Prove Equivalences using Symbolic Derivation

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

$$\begin{aligned}(p \wedge q)\rightarrow(p \vee q) &\Leftrightarrow \neg(p \wedge q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)\\ &\Leftrightarrow (\neg p \vee p) \vee (\neg q \vee q)\\ &\Leftrightarrow T \vee T\\ &\Leftrightarrow T\end{aligned}$$

Prove Equivalences using Symbolic Derivation

- HW:

Show that $\neg (p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q$

Prove Equivalences using Symbolic Derivation

- Check using a symbolic derivation whether $(p \wedge \neg q) \rightarrow (p \oplus r) \Leftrightarrow \neg p \vee q \vee \neg r$.

$(p \wedge \neg q) \rightarrow (p \oplus r)$ [Expand definition of $\rightarrow$]

$\Leftrightarrow \underline{\neg}(p \wedge \neg q)\ \underline{\vee}\ (p \oplus r)$ [Expand defn. of $\oplus$]

$\Leftrightarrow \neg(p \wedge \neg q) \vee (\underline{(p \vee r) \wedge \neg(p \wedge r)})$

[DeMorgan's Law]

$\Leftrightarrow \underline{(\neg p \vee q)} \vee ((p \vee r) \wedge \neg(p \wedge r))$

cont.

Example Continued...

$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \Leftrightarrow$ [$\vee$ commutes]

$\Leftrightarrow \underline{(q \vee \neg p)} \vee ((p \vee r) \wedge \neg(p \wedge r))$ [$\vee$ associative]

$\Leftrightarrow q \vee \underline{(}\neg p \vee ((p \vee r) \wedge \neg(p \wedge r)))$ [distrib. $\vee$ over $\wedge$]

$\Leftrightarrow q \vee (((\neg p \vee (p \vee r)) \wedge (\underline{\neg p \vee} \neg(p \wedge r)))$

[assoc.] $\Leftrightarrow q \vee (((\neg p \vee p) \vee r) \wedge (\neg p \vee \neg(p \wedge r)))$

[trivial taut.] $\Leftrightarrow q \vee ((\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r)))$

[domination] $\Leftrightarrow q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r)))$

[identity] $\Leftrightarrow q \vee (\neg p \vee \neg(p \wedge r)) \Leftrightarrow$ *cont.*

End of Long Example

$q \vee (\neg p \vee \neg(p \wedge r))$

[DeMorgan's] $\Leftrightarrow q \vee (\neg p \vee (\neg p \vee \neg r))$

[Assoc.] $\Leftrightarrow q \vee ((\neg p \vee \neg p) \vee \neg r)$

[Idempotent] $\Leftrightarrow q \vee (\neg p \vee \neg r)$

[Assoc.] $\Leftrightarrow (q \vee \neg p) \vee \neg r$

[Commut.] $\Leftrightarrow \neg p \vee q \vee \neg r$

Q.E.D. (quod erat demonstrandum)

(Which was to be shown.)