American University of Beirut
Dept. of Computer Science

# CMPS/Math 211

## Discrete Structures





# Section 2.3: Functions



## On to section 1.8… Functions

- From calculus, you are familiar with the concept of a real-valued function *f*, which assigns to each number $x \in \mathbf{R}$ a unique value $y=f(x)$, where $y \in \mathbf{R}$.
- A function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *mapping, a transformation.*)



## Function: Formal Definition

- For any sets $A$, $B$, we say that a *function f from (or "mapping") A to B* ($f:A \rightarrow B$) is a unique assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function $f$ assigns *zero or one* elements of $B$ to each element $x \in A$.
  - Functions of $n$ arguments.

## More

- A function $f: A \rightarrow B$ can also be defined in terms of a relation from $A$ to $B$, which is a subset of $A \times B$.
- A relation from $A$ to $B$ which contains one, and only one, ordered pair $(a,b)$ for every element $a$ in $A$ defines a function from $A$ to $B$.



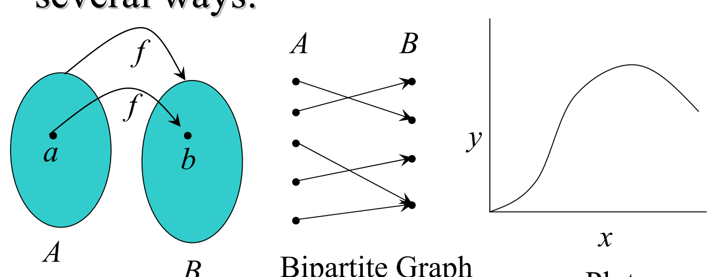
## More

- Functions are specified in various ways. Sometimes, we explicitly state the assignments.
- Often, we give a formula.
- Other times, we use a computer program to specify a function.



## Graphical Representations

- Functions can be represented graphically in several ways:




Like Venn diagrams | Bipartite Graph | Plot



## Functions We've Seen So Far

- A *proposition* can be viewed as a function from "situations" to truth values {**T**,**F**}
  - A logic system called *situation theory*.
  - $p$="It is raining."; $s$=our situation here,now
  - $p(s) \in \{\mathbf{T},\mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: *e.g.*, $\vee((\mathbf{F},\mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T},\mathbf{F})) = \mathbf{F}$.

## More functions so far…

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values): $P :\equiv$ "is 7 feet tall"; $P$(Mike) = "Mike is 7 feet tall." = **False**.
- A *bit string B of length n* can be viewed as a function from the numbers $\{1,\ldots,n\}$ (bit *positions*) to the *bits* {0,1}. *E.g.*, $B$=101 ➔ $B(3)$=1.



## Still More Functions

- A *set S* over universe *U* can be viewed as a function from the elements of *U* to {**T**, **F**}, saying for each element of *U* whether it is in *S*. $S=\{3\}$ ➔ $S(0)$=**F**, $S(3)$=**T**.
- A *set operator* such as $\cap, \cup, \overline{\phantom{x}}$ can be viewed as a function from pairs of sets to sets.
  - Example: $\cap((\{1,3\},\{3,4\})) = \{3\}$





## Terminology

- If it is written that $f:A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then we say:
  - *A* is the *domain* of *f*.
  - *B* is the *codomain* of *f*.
  - *b* is the *image* of *a* under *f*.
  - *a* is a *pre-image* of *b* under *f*.
    - In general, *b* may have more than 1 pre-image.
  - The *range* $R\subseteq B$ of *f* is $R=\{b \mid \exists a\, f(a)=b\}$.

We also say the *signature* of *f* is $A\rightarrow B$.



## Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

# Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}.





# Equal Functions

- Two functions are equal when they have the same domain, the same codomain, and map elements of their common domain to the same elements in their common codomain.
- If we change either the domain or the codomain of a function, we obtain a different function.
- If we change the mapping of elements, then we also obtain a different function.





# Examples

- Specify the functions embedded in the following definitions (to specify a function, we need to state the rule, domain, codomain, and range)
  - Let *R* be the relation consisting of ordered pairs (Abdul 22), (Brenda, 24), (Carla 21), (Desire, 22), (Eddie, 24), and (Felicia, 22), where each pair denotes a graduate student and corresponding age.
  - Let *f* be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example *f*(11010) = 10.
  - H.W. Examples 4 and 5 on page 135 of the 6th edition.



# Operators (general definition)

- An *n*-ary *operator over* (or *on*) the set *S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- *E.g.*, if $S=\{\mathbf{T},\mathbf{F}\}$, $\neg$ can be seen as a unary operator, and $\wedge,\vee$ are binary operators on *S*.
- Another example: $\cup$ and $\cap$ are binary operators on the set of all sets.

# Constructing Function Operators

- If • ("dot") is any operator over $B$, then we can extend • to also denote an operator over functions $f:A\rightarrow B$.
- *E.g.*: Given any binary operator $\bullet:B\times B\rightarrow B$, and functions $f,g:A\rightarrow B$, we define $(f \bullet g):A\rightarrow B$ to be the function defined by: $\forall a\in A$, $(f \bullet g)(a) = f(a)\bullet g(a)$.





# Function Operator Example

- $+,\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f,g:\mathbf{R}\rightarrow\mathbf{R}$:
  - $(f+g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f\times g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f\times g)(x) = f(x)\times g(x)$





# Images of Sets under Functions

- Given $f:A\rightarrow B$, and $S\subseteq A$,
- The *image* of $S$ under $f$ is simply the set of all images (under $f$) of the elements of $S$.

  $f(S) :\equiv \{f(s) \mid s\in S\}$

  $:\equiv \{b \mid \exists\, s\in S: f(s)=b\}$.
- Note the range of $f$ can be defined as simply the image (under $f$) of $f$'s domain!



# Example

- Let $f$ and $g$ denote functions from $\boldsymbol{R}$ to $\boldsymbol{R}$ such that $f(x) = x^2$ and $g(x) = x - x^2$ . What are the functions $f+g$ and $fg$?
- Let $A = \{a,b,c,d,e\}$, and $B = \{1,2, 3,4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. Find the image of the subset $S = \{b,c,d\}$.

# Function Composition Operator

Note match here.

- For functions $g$:$A\rightarrow B$ and $f$:$B\rightarrow C$, there is a special operator called *compose* ("○").
  - It composes (creates) a new function out of $f$ and $g$ by applying $f$ to the result of applying $g$.
  - We say $(f\circ g)$:$A\rightarrow C$, where $(f\circ g)(a) :\equiv f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that ○ (like Cartesian ×, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f\circ g \neq g\circ f$.)





# Function composition

- Note also that the composition $f\circ g$ cannot be defined unless the range of $g$ is a subset of the domain of $f$.
- When the composition of a function and its inverse is formed, in either order, an identity function is obtained (why?)
- Examples 20, 21.





# One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f$:$A\rightarrow B$, "$x$ is injective" :≡ $(\neg\exists x,y$: $x\neq y \wedge f(x)=f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
  - Domain & range have same cardinality. What about codomain?

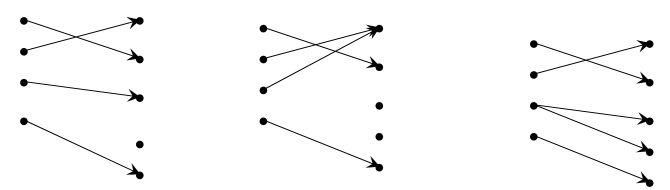
May Be Larger

- Memory jogger: Each element of the domain is injected into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."



# One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one

Not one-to-one

Not even a function!

# Exercises

- Examples 8 – 10.



# Sufficient Conditions for injectivity

- For functions $f$ over numbers, we say:
  - $f$ is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all $x,y$ in domain;
  - $f$ is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all $x,y$ in domain;
- If $f$ is either strictly increasing or strictly decreasing, then $f$ is one-to-one. *E.g.* $x^3$
  - *Converse is not necessarily true. E.g.* $1/x$





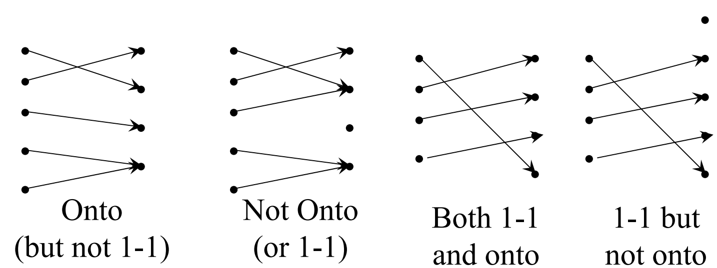
# Onto (Surjective) Functions

- A function $f{:}A\rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b\in B, \exists a\in A{:}\ f(a)=b$).
- Think: An *onto* function maps the set $A$ onto (over, covering) the *entirety* of the set $B$, not just over a piece of it.
- *E.g.*, for domain & codomain $\mathbf{R}$, $x^3$ is onto, whereas $x^2$ isn't. (Why not?)



# Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



# Exercises

- Examples 11– 13.



# Bijections

- A function $f$ is said to be *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f:A\to B$, there exists an *inverse of* $f$, written $f^{-1}:B\to A$, which is the unique function such that $f^{-1}\circ f=I_A$
  - (where $I_A$ is the identity function on $A$)





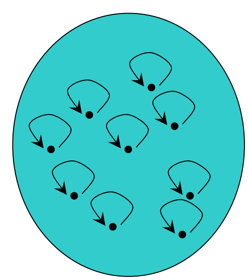
# The Identity Function

- For any domain $A$, the *identity function* $I:A\to A$ (variously written, $I_A$, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a\in A: I(a)=a$.
- Some identity functions you've seen:
  - $+$ing 0, $\cdot$ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with $U$.
- Note that the identity function is always both one-to-one and onto (bijective).

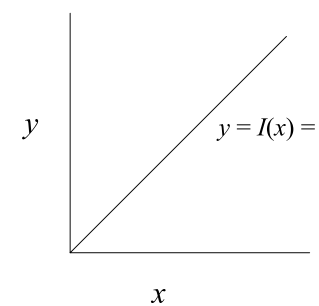


# Identity Function Illustrations

- The identity function:



Domain and range



# Bijections

- Suppose that $f$ is a function from a set $A$ to itself. If $A$ is finite, then $f$ is one-to-one if and only if it is onto.
- This is not the case if $A$ were infinite.
- Examples 16—19.



# Graphs of Functions

- We can represent a function $f{:}A\rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a\in A\}$. ← The function's *graph*.
- Note that $\forall a$, there is only 1 pair $(a,b)$.
  - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair $(x,y)$ as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one $y$ for each $x$.





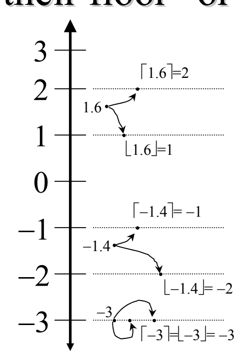
# A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
  - The *floor* function $\lfloor\cdot\rfloor{:}\mathbf{R}\rightarrow\mathbf{Z}$, where $\lfloor x\rfloor$ ("floor of $x$") means the largest (most positive) integer $\leq x$. *I.e.*, $\lfloor x\rfloor :\equiv \max(\{i\in\mathbf{Z} \mid i\leq x\})$.
  - The *ceiling* function $\lceil\cdot\rceil{:}\mathbf{R}\rightarrow\mathbf{Z}$, where $\lceil x\rceil$ ("ceiling of $x$") means the smallest (most negative) integer $\geq x$. $\lfloor x\rfloor :\equiv \min(\{i\in\mathbf{Z} \mid i\geq x\})$

Module #4 - Functions

# Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x\notin\mathbf{Z}$, $\lfloor -x\rfloor \neq -\lfloor x\rfloor$ & $\lceil -x\rceil \neq -\lceil x\rceil$
- Note that if $x\in\mathbf{Z}$, $\lfloor x\rfloor = \lceil x\rceil = x$.



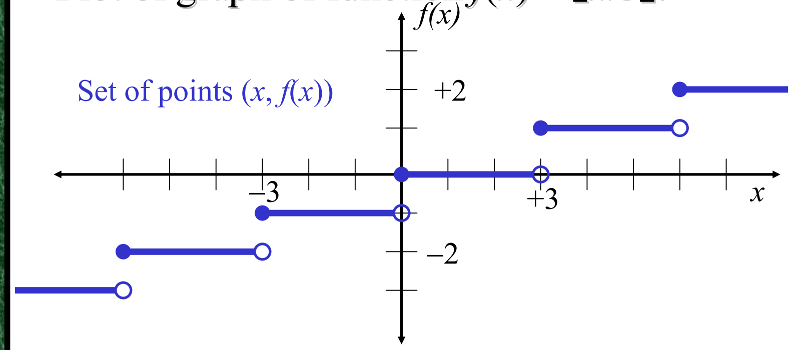
# Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of $f$ includes the point $(a, 0)$ for all values of $a$ such that $a\geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in $f$ does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.



# Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:








# Exercises

- Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?
- Examples 28, 29.
- H.W. 26.
- Look up Table 1 on page 144 of the 6th edition. Make sure you know it well.