American University of Beirut
Dept. of Computer Science

CMPS/Math 211

Discrete Structures

Lecture 4

Section 1.4: Nested Quantifiers

Nesting of Quantifiers (1.4)

Example: Let the u.d. of x & y be people.

Let $L(x,y)$="x likes y" (a predicate w. 2 f.v.'s)

Then $\exists y\, L(x,y)$ = "There is someone whom x likes." (A predicate w. 1 free variable, x)

Then $\forall x\, (\exists y\, L(x,y))$ =
"Everyone has someone whom they like."
(A **Proposition** with 0 free variables.)

- What do the following statements mean, where the domain consists of all real numbers?
- $\forall x\, \forall y\, (x+y = y+x)$
- $\forall x\, \exists y\, (x+y = 0)$
- $\forall x\, \forall y\, \forall z\, (x+ (y + z) = (x + y) + z))$
- $\forall x\, \forall y\, ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\, R(x,y))$= Everyone has *someone* to rely on.

$\exists y(\forall x\, R(x,y))$= There's an overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, R(x,y))$= There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, R(x,y))$= Everyone has *someone* who relies upon them.

$\forall x(\forall y\, R(x,y))$= *Everyone* relies upon *everybody*, (including themselves)!

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like,
    - $\forall x\, \exists y\, Likes(x,y)$ [Probably more likely.]
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y\, \forall x\, Likes(x,y)$
- "Somebody likes everybody."
  - Same problem: Depends on context, emphasis.

Still More Conventions

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for "For all x that are greater than zero, $P(x)$." $=\forall x\,(x>0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for "There is an x greater than zero such that $P(x)$." $=\exists x\,(x>0 \land P(x))$

More to Know About Binding

- $\forall x\, \exists x\, P(x)$ - x is not a free variable in $\exists x\, P(x)$, therefore the $\forall x$ binding <u>isn't used</u>.
- $(\forall x\, P(x)) \land Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!
- $(\forall x\, P(x)) \land (\exists x\, Q(x))$ – This is legal, because there are 2 <u>different</u> x's!

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\ P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\ P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\ P(x) \Leftrightarrow \neg\exists x\ \neg P(x)$
  $\exists x\ P(x) \Leftrightarrow \neg\forall x\ \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this? **DeMorgan's**

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$
- Exercise:
  See if you can prove these yourself.
  - What propositional equivalences did you use?

More Notational Conventions

- Quantifiers bind as loosely as needed: parenthesize $\forall x (P(x) \wedge Q(x))$
- Consecutive quantifiers of the same type can be combined: $\forall x\ \forall y\ \forall z\ P(x,y,z) \Leftrightarrow \forall x,y,z\ P(x,y,z)$ or even $\forall xyz\ P(x,y,z)$
- All quantified expressions can be reduced to the canonical *alternating* form $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots\ P(x_1, x_2, x_3, x_4, \ldots)$

- Examples 4, 5, 6, 7, and10 from book.

- Express the following statements as a logical expression involving predicates, quantifiers with a domain consisting of people, and logical connectives.
  - ``If a person is female and is a parent, then this person is someone's mother''
  - Everyone has exactly one friend
  - There is a woman who has taken a flight on every airline in the world (take u.d. to be all the women in the world)
  - There does not exists a woman who has taken a flight on every airline in the world.