American University of Beirut
Dept. of Computer Science

CMPS/Math 211

Discrete Structures

Lecture 4

Section 1.3: Predicates and Quantifiers

Predicate Logic (§1.3)

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.
- Propositional logic (recall) treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate* logic distinguishes the *subject* of a sentence from its *predicate*.

Applications of Predicate Logic

It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* (more on these in module 2) for *any* branch of mathematics.

Predicate logic with function symbols, the "=" operator, and a few proof-building rules is sufficient for defining *any* conceivable mathematical system, and for proving anything that can be proved within that system!

Practical Applications of Predicate Logic

- It is the basis for clearly expressed formal specifications for any complex system.
- It is basis for *automatic theorem provers* and many other Artificial Intelligence systems.
  - *E.g.* automatic program verification systems.

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

Universes of Discourse (U.D.s)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* subjects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \wedge (\mathbf{1}+1>\mathbf{1}) \wedge (\mathbf{2}+1>\mathbf{2}) \wedge \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse*.

More About Predicates

- Convention: Lowercase variables x, y, z... denote subjects; uppercase variables P, Q, R… denote propositional functions (predicates).
- Keep in mind that the *result of applying* a predicate P to an object x is the *proposition* $P(x)$. But the predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).
  - *E.g.* if $P(x)$ = "x is a prime number", $P(3)$ is the *proposition* "3 is a prime number."

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments.
  - *E.g.* let $P(x,y,z)$ = "x gave y the grade z", then if x="Mike", y="Mary", z="A", then $P(x,y,z)$ = "Mike gave Mary the grade A."

Exercises

- Let $P(x)$ denote the statement `$x > 3$'. What are the truth values of $P(4)$ and $P(2)$?
- Let $Q(x,y)$ denote the statement ``x = y+3''. What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

Quantifier Expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of disc. satisfy a given predicate.
- "$\forall$" = ``FOR ALL'' or *universal* quantifier. $\forall x\, P(x)$ means *for all* x in the u.d., *P* holds.
- "$\exists$" = ``there exists'' or *existential* quantifier. $\exists x\, P(x)$ means there *exists* an x in the u.d. (that is, one or more) such that $P(x)$ is true.

The Universal Quantifier $\forall$

- Example:
  Let the u.d. of x be parking spaces at UF.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at UF are full."
  - *i.e.*, "Every parking space at UF is full."
  - *i.e.*, "For each parking space at UF, that space is full."

Exercises

- Let *P(x)* denote the statement ``$x+1>x$''. What is the truth value of the quantification $\forall x\, P(x)$ where the domain consists of all real numbers?
- Same question for the statement *Q(x)* corresponding to ``$x < 2$''.
- Same question for the statement *P(x)* corresponding to `` $x^2 < 10$'' and the domain consists of the positive integers not exceeding 4

Exercise

- What is the truth value of ($x^2 \geq x$) if the domain consists of all real numbers? If the domain consists of all integers?

The Existential Quantifier ∃

- Example:
  Let the u.d. of x be parking spaces at UF.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at UF is full."
  - "There is a parking space at UF that is full."
  - "At least one parking space at UF is full."

Exercises

- Let *P(x)* denote the statement ``$x>3$''. What is the truth value of the quantification $\exists x\, P(x)$ where the domain consists of all real numbers?
- Same question when *P(x)* denotes the statement ``$x = x + 1$''.

Exercises

- Same question for the statement *P(x)* corresponding to `` $x^2 > 10$'' and the domain consists of the positive integers not exceeding 4

Quantifiers with restricted domains

- What do the following statements mean, where the domain in each case consists of the real numbers?
- $\forall x < 0\ (x^2 > 0)$
- $\forall y \neq 0 (y^3 > 0)$
- $\exists z > 0\ (z^2 = 2)$

Precedence of Quantifiers

- The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus.
- Example, how do you evaluate

$$\forall x\ P(x) \vee Q(x)?$$

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning, x is undefined).
- A quantifier (either ∀ or ∃) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable, and one bound variable. [Which is which?]
- "$P(x)$, where x=3" is another way to bind x.
- An expression with <u>one or more</u> free variables is still only a predicate: *e.g.* let $Q(y) = \forall x\, P(x,y)$

Logical Equivalences Involving Quantifiers

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

Exercise

- Show that $\forall\, x\, (P(x) \wedge Q(x))$ and $\forall\, x\, P(x) \wedge \forall\, x\;\; Q(x)$ are logically equivalent, where the same domain is used throughout (A is true iff B is true).

Negating Quantified Expressions

- Find the negation of ``*Every student in your class has taken a course in calculus*''.
- We have the following two logical equivalences, also denoted by De Morgan's laws for quantifiers:

  $\neg\forall x\, P(x)$ is equivalent to $\exists\, \neg\, P(x)$

  $\neg\, \exists\, x\, Q(x)$ is equivalent to $\forall\, x\, \neg Q(x)$

De Morgan's Laws for Quantifiers

$\neg\forall x\ P(x)$ is equivalent to $\exists\ \neg\ P(x)$

When is Negation True?

For every x, $P(x)$ is false.

When is Negation False?

There is an x for which $P(x)$ is true.

De Morgan's Laws for Quantifiers

$\neg\ \exists\ x\ P(x)$ is equivalent to $\forall\ x\ \neg P(x)$

When is Negation True?

There is an x for which $P(x)$ is false.

When is Negation False?

$P(x)$ is true for every x.

Exercises

- What are the negations of the statements ``There is an honest politician'' and ``All Americans eat cheeseburgers''?
- What are the negations of the statements

$\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$

- Show that $\neg\forall\ xP(x)$ and $\exists\ x\ (P(x) \wedge \neg Q(x))$ are logically equivalent.
- H.W. Prove De Morgan's laws for quantifiers (p.40 of your book).

Translating from English

Express the following statements using predicates and quantifiers:

- ``Every student in this class has studied calculus''.
- Some student in this class has visited Mexico.
- Every student in this class has visited either Canada or Mexico.