CMPS 211
Fall 2010

Assignment II - Solution

Section 4.1

4- $P(n)$: $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

a) $P(1)$: $1^3 = \left(\frac{1(1+1)}{2}\right)^2$

b) $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1 = 1^3$ $\Rightarrow$ $P(1)$ is true.

c) Inductive Hypothesis: $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

d) Assuming the inductive hypothesis in (c), need to show that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

e) $1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{by inductive hypothesis}$$

$$= (n+1)^2\left[\left(\frac{n}{2}\right)^2 + (n+1)\right] = (n+1)^2\left(\frac{n^2+4n+4}{2^2}\right)$$

$$= (n+1)^2\left(\frac{(n+2)^2}{2^2}\right) = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

f) By completing the basis step and the inductive step, we've shown that this statement is true for all positive integers *n*.

6- $1\cdot1! + 2\cdot2! + \ldots + n\cdot n! = (n+1)! - 1$

Basis step: $1\cdot1! = 1$ $\quad (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

$\Rightarrow 1\cdot1! = (1+1)! - 1.$

Inductive Hypothesis: $1\cdot1! + 2\cdot2! + \ldots + n\cdot n! = (n+1)! - 1$

Inductive Step: Show that $1\cdot1! + 2\cdot2! + \ldots + n\cdot n! + (n+1)\cdot(n+1)! = (n+2)! - 1$?

$$1\cdot1! + 2\cdot2! + \ldots + n\cdot n! + (n+1)\cdot(n+1)! = (n+1)! - 1 + (n+1)\cdot(n+1)!$$

$$= (n+1)!\,[1+(n+1)] - 1$$

$$= (n+1)!\,(n+2) - 1 \quad = (n+2)! - 1$$

10- a) Formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}$

For n = 1: $\frac{1}{1\cdot 2}=\frac{1}{2}$

For n = 2: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$

For n = 3: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$

For positive integers n: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) Show $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Basis Step: *shown in part (a) above for n = 1*

Inductive Hypothesis: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Inductive Step: Show $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)} = \frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

$$= \frac{n\,(n+2)+1}{(n+1)(n+2)} \qquad = \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)} \qquad = \frac{n+1}{n+2}$$

16 - 1·2·3 + 2·3·4 +...+ n(n+1)(n+2)= $\frac{n(n+1)(n+2)(n+3)}{4}$

Basis Step: true for n=1, P(1) true, 1·2·3=$\frac{1(1+1)(1+2)(1+3)}{4}$, 6 = 6

Inductive Step: Suppose that P(k) is true, k > 0

1·2·3+ 2·3·4+...+ k(k+1)(k+2)= $\frac{k(k+1)(k+2)(k+3)}{4}$

Show that P(k+1) is true

1·2·3+ 2·3·4+...+ k(k+1)(k+2) + (k+1)(k+2)(k+3)

=$\frac{k(k+1)(k+2)(k+3)}{4}$ + (k+1)(k+2)(k+3) (by inductive hypothesis)

=(k+1)(k+2)(k+3)$\left[\frac{k}{4}+1\right]$

=(k+1)(k+2)(k+3) $\left[\frac{k+4}{4}\right]$

$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$ as required.

Therefore, P(k+1) true.

And $1\cdot2\cdot3+2\cdot3\cdot4+\ldots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$, n>0.

22. We will show using mathematical induction that $n^2 \leq n!$ for all $n \geq 4$ since it is not true for $n = 2$, and $n = 3$ but true for $n = 0$ and $n = 1$

Basis Step: $(n = 4)$
$4^2 = 16 \leq 24 = 4!$
$\Rightarrow$ True for $n = 4$

Inductive Hypothesis: Suppose the statement is true for $n = k - 1$
i.e. $(k-1)^2 \leq (k-1)!$ for $(k-1) \geq 4$

Inductive Step: Show that statement is true for $n = k$
i.e. $k^2 \leq k!$

$(k-1)^2 \leq (k-1)!$ (by inductive hypothesis)
$k^2 - 2k + 1 \leq (k-1)!$
$\Rightarrow k^2 - 2k \leq (k-1)!$
$\Rightarrow k(k-2) \leq (k-1)!$
$\Rightarrow k^2(k-2) \leq k(k-1)$ (multiplying both sides by $k > 1$)
$\Rightarrow k^2(k-2) \leq k!$
$\Rightarrow k^2 \leq k!$ (since $(k-2) > 1$) as required.

Therefore, $n^2 \leq n!$ for all $n \geq 4$.

32- Prove that 3 divides n^3+2n for all $n>0$.

Basis Step: (n=1) $1^3+2 = 1+2 = 3$ divisible by 3
$\Rightarrow$ True for n=1.

Inductive Hypothesis: n^3+2n is divisible by 3.

Inductive Step: Show that $(n+1)^3+2(n+1)$ is divisible by 3.

$$(n+1)^3+2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2$$
$$= n^3 + 3n^2 + 5n + 3$$
$$= (n^3 + 2n) + 3n^2 + 3n + 3$$
$$= (n^3 + 2n) + 3(n^2+n+1)$$

But $(n^3 + 2n)$ is divisible by 3 by inductive hypothesis
and $3(n^2+n+1)$ is divisible by 3 multiplied by 3
So, $(n^3 + 2n)+ 3(n^2+n+1)$ is divisible by 3.
$\Rightarrow (n+1)^3+2(n+1)$ is divisible by 3
$\therefore$ 3 divides n^3+2n for all $n>0$

Section 4.2

4- $P(n)$: a postage of n cents can be formed using 4-cent stamps and 7-cent stamps

Show $P(n)$ is true for all $n \geq 18$.

a) $P(18)$: $18 = 4+14 = 4\cdot1 + 7\cdot2$ $\Rightarrow$ True for n=18

$P(19)$: $18 = 12+7 = 4\cdot3 + 7\cdot1$ $\Rightarrow$ True for n=19

$P(20)$: $20 = \quad = 4\cdot5 + 7\cdot0$ $\Rightarrow$ True for n=20

$P(21)$: $21 = \quad = 4\cdot0 + 7\cdot3$ $\Rightarrow$ True for n=21

b) Inductive Hypothesis: $P(n)$ is true for all $18 \leq n \leq k$,

i.e. $P(n) = 4k_1 + 7k_2$ for some k_1 , $k_2 \geq 0$

c) Inductive Step: Required to prove that $P(k+1)$ is true for $k \geq 22$.

d) $P(k+1) = P(k) + 1$

$= P(k–1) + 2$

$= P(k–2) + 3$

$= P(k–3) + 4$

But $P(k–3)$ is true by inductive hypothesis

i.e. $P(k–3) = 4k_1 + 7k_2$ for some k_1 , $k_2 \geq 0$

So $P(k+1) = 4k_1 + 7k_2 + 4 = 4(k_1 +1) + 7k_2$

$\Rightarrow P(k+1)$ is true for all $k \geq 22$

$\therefore$ $P(n)$ is true for all $n \geq 18$.

8.

Possible amounts that can be formed using these gift certificates are:

25$ certificates	40$ certificates	Total
1	0	25$
0	1	40$
1	1	65$
3	0	75$
0	2	80$
2	1	90$
4	0	100$
1	2	105$
3	1	115$
0	3	120$
5	0	125$
2	2	130$
4	1	140$
1	3	145$
6	0	150$
3	2	155$
0	4	160$

We claim that every amount divisible by 5 and above 140$ is available.
Let P(n) be the statement that the amounts in the table above and every amount divisible by 5 from 140$ up to 140$+5n$ can be formed from the gift certificates.

The basis step is proved in the table above, all possible quantities under 160$ were examined and the statement proven true for these values.

Inductive Hypothesis: Suppose that P is true for all numbers up to n, where n > 4, (P(0), P(1), P(2), P(3),and P(4)) are proven true.

For n + 1, we need to show that 140 + 5(n + 1) can be formed from 25 and 40 dollar certificates.
n+1 > 5 so 140+5(n+1) = 140+5n+5 = 140+5(n-4)+20+5 = 140 + 5(n - 4) + 25 but 140 + 5(n - 4) is P(n - 4) which is true by inductive hypothesis and can be formed of k_1 gift certificates of the 25 dollars and k_2 gift certificates of the 40 dollars. So, the new quantity resembling P(n+1) can be formed of (k_1 +1) gift certificates of the 25 dollars and k_2 gift certificates of the 40 dollars. Thus, P(n+1) is true.

Therefore, P(n) is true for all the values claimed above.

Section 4.3

4- a) $f(n+1) = f(n) - f(n-1)$ $\quad f(0) = f(1) = 1$

$f(2) = f(1) - f(0) = 1 - 1 = 0$
$f(3) = f(2) - f(1) = 0 - 1 = -1$
$f(4) = f(3) - f(2) = (-1) - 0 = -1$
$f(5) = f(4) - f(3) = (-1) - (-1) = 0$

b) $f(n+1) = f(n)\, f(n-1)$ $\quad f(0) = f(1) = 1$

$f(2) = f(1)\, f(0) = (1)\,(1) = 1$
$f(3) = f(2)\, f(1) = (1)\,(1) = 1$
$f(4) = f(3)\, f(2) = (1)\,(1) = 1$
$f(5) = f(4)\, f(3) = (1)\,(1) = 1$

c) $f(n+1) = f(n)^2 + f(n-1)^3$ $\quad f(0) = f(1) = 1$

$f(2) = f(1)^2 + f(0)^3 = (1)^2 + (1)^3 = 1 + 1 = 2$
$f(3) = f(2)^2 + f(1)^3 = (2)^2 + (1)^3 = 4 + 1 = 5$
$f(4) = f(3)^2 + f(2)^3 = (5)^2 + (2)^3 = 25 + 8 = 33$
$f(5) = f(4)^2 + f(3)^3 = (33)^2 + (5)^3 = 1089 + 125 = 1214$

d) $f(n+1) = f(n) / f(n-1)$ $\quad f(0) = f(1) = 1$

$f(2) = f(1) / f(0) = 1 / 1 = 1$
$f(3) = f(2) / f(1) = 1 / 1 = 1$
$f(4) = f(3) / f(2) = 1 / 1 = 1$
$f(5) = f(4) / f(3) = 1 / 1 = 1$

6- a) $f(0) = 1$, $\quad f(n) = -\, f(n-1)$ for $n \geq 1$

Valid: $\quad f(n) = (-1)^n$ for $n \geq 1$

Proof: *Basis Step*: $f(1) = (-1)^1 = -1$

Inductive Hypothesis: $f(n) = (-1)^n$ for $n \geq 1$

Inductive Step: $f(n+1) = -\, f(n) = -(-1)^n = (-1)(-1)^n = (-1)^{n+1}$

d) $f(0) = 0, f(1) = 1$ $\quad f(n) = 2\, f(n-1)$ for $n \geq 1$

Not Valid: defined starting at n=1, but $f(1) = 2\, f(0) = 2 \cdot 0 = 0 \neq 1$.

e) $f(0) = 2$, $\quad f(n) = f(n-1)$ if n is odd and $n \geq 1$

$f(n) = 2\, f(n-2)$ if $n \geq 2$

Valid: $$f(n) = \begin{cases} 2^{(n+2)/2} & \text{if } n \text{ is even and } n \geq 0 \\ 2^{(n+1)/2} & \text{if } n \text{ is odd and } n \geq 0 \end{cases}$$

Proof: *Basis Step*: $f(0) = 2^{(0+2)/2} = 2^1 = 2$

$f(1) = 2^{(1+1)/2} = 2^1 = 2 = f(0)$

Inductive Hypothesis: $$f(n) = \begin{cases} 2^{(n+2)/2} & \text{if } n \text{ is even and } n \geq 0 \\ 2^{(n+1)/2} & \text{if } n \text{ is odd and } n \geq 0 \end{cases}$$

Inductive Step:

- n is even $\Rightarrow$ $f(n) = 2^{(n+2)/2} \Rightarrow n+1$ is odd
  $\Rightarrow$ Show $f(n+1) = 2^{(n+1+1)/2} = 2^{(n+2)/2}$
  $f(n+1) = f((n+1)-1)$ (by recursive definition)
  $= f(n) = 2^{(n+2)/2}$ (by induction hypothesis) **True**
- n is odd $\Rightarrow$ $f(n) = 2^{(n+1)/2} \Rightarrow n+1$ is even
  $\Rightarrow$ Show $f(n+1) = 2^{(n+1+2)/2} = 2^{(n+3)/2}$
  $f(n+1) = 2\, f((n+1)-1)$ (by recursive definition)
  $= 2\, f(n) = 2 \cdot 2^{(n+1)/2}$ (by induction hypothesis)
  $= 2^{(n+1)/2+1} = 2^{(n+1+2)/2} = 2^{(n+3)/2}$ **True**

10- $S_m(n)$ is the sum of the integer m and the nonnegative integer n.
$S_m(0) = m$, $S_m(1) = m+1$, $S_m(2) = m+2 = (m+1)+1, \ldots$
Recursive Definition: $S_m(0) = m$, $S_m(n) = S_m(n-1) + 1$ for $n \geq 1$.

12- f_n is the nth Fibonacci number.
Show $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ (using induction).
Let $P(n)$ be the statement "$f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$".
Basis Step: $P(1)$ is true because $f_1^2 = f_1 f_1 = f_1 f_2$ since $f_1 = f_2 = 1$.
Inductive Hypothesis: Assume $P(n)$ is true, i.e. $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$
Inductive Step: Show that $P(n+1)$ is true, i.e. $f_1^2 + f_2^2 + \ldots + f_n^2 + f_{n+1}^2 = f_{n+1} f_{n+2}$

$P(n+1) = f_1^2 + f_2^2 + \ldots + f_n^2 + f_{n+1}^2$
$= P(n) + f_{n+1}^2$
$= f_n f_{n+1} + f_{n+1}^2$ (by inductive hypothesis)
$= f_{n+1} (f_n + f_{n+1})$
$= f_{n+1}\ f_{n+2}$ (by Fibonacci formula)

Thus, $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ is true for all $n \geq 1$.

22 - Let A be the set of positive integers. We will show that S = A. We will show that A $\subseteq$ S and S $\subseteq$ A.

We will first show that A is a subset of S.
Let P(n) be the statement that n belongs to S, where n is an element in A.

Basis Step: n = 1, true since 1 $\in$ S

Hypothesis: Suppose P(k) is true, i.e. k $\in$ S

Inductive Step: Show that P(k + 1) is true.

P(k + 1) is true if (k + 1) $\in$ S, but k + 1 = (k) + (1), where k $\in$ S and
1 $\in$ S and thus (k + 1) $\in$ S.
P(n) is true and A $\subseteq$ A.

To show that S $\subseteq$ A, we'll show that the base step of S is in A and elements generated by the recursive formula are in A as well, i.e. (s + t) $\in$ A whenever s $\in$ S and t $\in$ S.
1 $\in$ A, since 1 is a positive integer.

Let s, t $\in$ S; s and t are positive integers and their sum s + t = k is positive. Since k is a positive integer, then k $\in$ A, (A is the set of all positive integers).

Therfeore, A = S.

24- a) the set of odd positive integers: set *S*
$1 \in S$, and if $x \in S$ then $x+2 \in S$.

b) the set of positive integer powers of 3: set *S*
$1=3^0 \in S$, and if $x \in S$ then $3x \in S$

c) the set of polynomials with integer coefficients: set *S*
$c_0\, x^0 \in S$, and if $c_0\, x^0 + c_1\, x^1 + \ldots + c_n\, x^n \in S$
then $c_0\, x^0 + c_1\, x^1 + \ldots + c_n\, x^n + c_{n+1}\, x^{n+1} \in S$
where $c_i \in$ N, $i \geq 0$

26- *S* is the subset of the set of ordered pairs of integers defined recursively by
Basis Step: $(0,0) \in S$.
Recursive Step: If $(a,b) \in S$, then $(a+2,b+3) \in S$ and $(a+3,b+2) \in S$.

a) (2,3) , (3,2) ; (4,6) , (5,5) , (6, 4) ; (6,9) , (7,8) , (8,7) , (9,6); (8,12) , (9,11) , (10,10) , (11,9) , (12,8) ; (10,15) , (11,14) , (12,13) , (13,12) , (14,11) , (15,10).

b) Let *P(n)* be the statement that 5 divides $a+b$ whenever $(a,b) \in S$ is obtained by *n* applications of the recursive step.
Basis Step: *P(0)* is true because 5 divides 0+0 and $(0,0) \in S$ obtained with 0 steps of the recursion.
Inductive Step: Assume *P(k)* is true, i.e. 5 divides $a+b$ whenever $(a,b) \in S$ is obtained by *k* or less applications of the recursive step.
Show *P(k+1)* is true, i.e. it's valid for an element obtained with *k+1* applications of the recursion.
So 5 divides $a+b$ (which is obtained by fewer iterations), and applying the final recursive step results in $(a+2,b+3)$ and $(a+3,b+2)$.
Case 1: $a+2 + b+3 = a+b +5 = (a+b) +5$ divisible by 5.
Case 1: $a+3 + b+2 = a+b +5 = (a+b) +5$ divisible by 5.

32- a) *ones(s)* : counts the number of ones in a bit string *s*.
Let *x* be a single bit, i.e. $x \in \{0,1\}$. Let *s* be a bit string, i.e. $s \in \{0,1\}^*$.

$$ones(x) = \begin{cases} 1 & if\ \ x = 1 \\ 0 & if\ \ x = 1 \end{cases}$$

Then, $s = tx$ where t is also a bit string, and $|t| = |s|-1$.
So, $ones(s) = ones(t\,x) = ones(t) + ones(x)$.

50- $A(1,n) = 2^n$ for $n \geq 1$.
Let $P(n)$ be the statement "$A(1,n) = 2^n$" for $n \geq 1$.
Basis Step: $P(1)$ is true because $A(1,1) = 2$ according to Ackermann's function
and $A(1,1) = 2^1 = 2$ according to $P(1)$

Inductive Step: Assume $P(k)$ is true for $k \geq 1$, i.e. $A(1,k) = 2^k$
Show that $P(k+1)$ is true, i.e. $A(1,k+1) = 2^{k+1}$

$A(1,k+1) = A(0, A(1,k))$ according to Ackermann's function – rule 4
$= A(0,2^k)$ according to induction hypothesis
$= 2\,(2^k)$ according to Ackermann's function – rule 1
$= 2^{k+1}$

Thus, using the basis step and the inductive step, and by the principle of mathematical induction, $A(1,n) = 2^n$ for $n \geq 1$.

Section 3.1

4- **procedure** *maxDiff* ($a_1, a_2, \ldots, a_n$: integers)
diff := 0
maxDiff := 0
for i := 1 to n-1
  diff := $a_i - a_{i+1}$
  if *absValue* (diff) > maxDiff
    then maxDiff := *absValue* (diff)
{maxDiff has the desired value}

6- **procedure** *numNegative* ($a_1, a_2, \ldots, a_n$: integers)
count := 0
for i := 1 to n
  if $a_i < 0$ **then** count := count + 1
{count has the desired value}

10- **procedure** *powerNonNeg* (x: real; n: positive integer)
power := 1
for i := 1 to n
  power := power × x
{power has the desired value, and this is a subprocedure being called by another}

procedure *power* (x:real; n: integer)
power := *powerNonNeg* (x, *absValue(n)*)
if $n < 0$ **then** power := 1 / power

{power has the desired value, for all negative and nonnegative powers}

18- **procedure** *lastSmallest* ($a_1, a_2, \ldots, a_n$: integers)
smallest := a_1
smallestLoc := 1
for i := 2 to n
  if $a_i \leq$ smallest **then**
  begin
    smallest := a_i
    smallestLoc := i
  end
{smallestLoc has the desired value}

32- **procedure** *greaterTerms* ($s_1, s_2, \ldots, s_n$: integers)
sum := s_1
termList := {}
for i := 2 to n
  if a_i > sum **then** termList := termList $\cup$ s_i
  sum := sum + s_i
{termList has the desired set}

34 - The lists obtained at each step are:

$i = 1; j = 1; (2; 6; 3; 1; 5; 4)$

$i = 1; j = 2; (2; 3; 6; 1; 5; 4)$

$i = 1; j = 3; (2; 3; 1; 6; 5; 4)$

$i = 1; j = 4; (2; 3; 1; 5; 6; 4)$

$i = 1; j = 5; (2; 3; 1; 5; 4; 6)$

$i = 2; j = 1; (2; 3; 1; 5; 4; 6)$

$i = 2; j = 2; (2; 1; 3; 5; 4; 6)$

$i = 2; j = 3; (2; 1; 3; 5; 4; 6)$

$i = 2; j = 4; (2; 1; 3; 4; 5; 6)$

$i = 3; j = 1; (1; 2; 3; 4; 5; 6)$

$i = 3; j = 2; (1; 2; 3; 4; 5; 6)$

$i = 3; j = 3; (1; 2; 3; 4; 5; 6)$

$i = 4; j = 1; (1; 2; 3; 4; 5; 6)$

$i = 4; j = 2; (1; 2; 3; 4; 5; 6)$

$i = 5; j = 1; (1; 2; 3; 4; 5; 6)$